Inventory Control – subject to known demand

Production and operation analysis

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Types of Inventory

- Raw materials,
- Components; items which are not yet reached completion in the production system such as subassemblies,
- Work in process (work in progress); is inventory waiting in the system to be processed or being processed,
- Finished goods; final products which completed all the processes.

Motivation for Inventory

- It improves system performance by decoupling parts of the system from one another,
- It allows production system to be built with capacity less than the peak demand,
- It reduces the propagation of disturbances and thus reduces instability and fragility of complex, expensive systems,

Motivation for Inventory

- Economies of scale; it might be economical to produce or to order in large amounts and store for future use.
- Uncertainties; Demand changes, lead time variability, supply chain uncertainties etc..
- Speculation; inventory may be held in anticipation of a rise in their value or cost.
- Transportation; in transit inventories which are applicable for pipeline industries.
- Smoothing; to resist against demand fluctuations

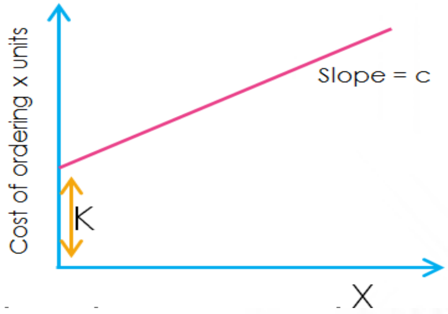
Some terminology

- Lead time refers to time between the arrival of a order and the accomplishment of it. For example;
  - Order lead time; time between the order given to the supplier and the arrival of it to the company
  - Production lead time; time from the start of a production order to the completion.
- Review time; intervals of time for controlling of inventories. E.g. Periodic review, Continuous review (in supermarkets through barcode system)
- Excess demand; when demand cannot be met, they are either backlogged(delivered later) or lost.
- Changing inventory; time might affect the utility. Short shelf life, becoming obsolete etc…

Relevant costs

- Holding cost,
  - Cost of providing the physical space to store items,
  - Taxes and insurance,
  - Breakage, spoilage, deterioration, obsolescence,
  - Opportunity cost.
- Order cost, is related with the amount inventory ordered and the number of occasions.

$$C(x)=\begin{cases} 0 & if\ x = 0 \\ K + cx & if x > 0 \end{cases}$$



- Penalty cost refers to shortage cost or stockout cost

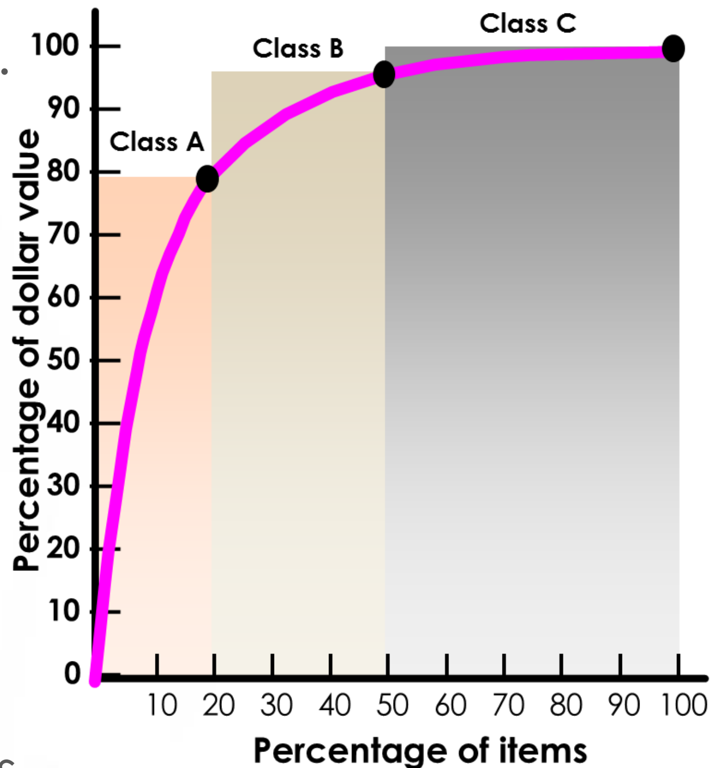
ABC inventory classification method

Multiproduct systems

- Pareto effect in inventory management: a large portion of the total value of sales is often accounted for by a small number of inventory items.
- Typically, the top 20% of the items (class A) account for about 80% of the annual sales value, the next 30% of the items (class B) for the next 15% of sales and the remaining 50% (class C) is for the last 5% of sales value.
  - Class A- high annual dollar volume
  - Class B-medium annual dollar volume
  - Class C-low annual dollar volume

ABC analysis

- Tight management control of ordering procedures is essential for *Type A* items.
- For *Type B* items, inventories can be reviewed periodically
  - Items can be ordered in small groups, rather than individually.
- *Type C* items require the minimum degree of control
  - Parameters are reviewed twice a year. Demand for Type C items may be forecasted by simple methods. The most inexpensive items of type C can be ordered in large lot, to minimize number of orders. An expensive type C items ordered only as they are demanded.



Fundemantel questions of inventory control

1. What items to stock?

   Objectives of a business and the strategy to achieve the objectives. e.g. range of stock offered by retailers

2. Where to stock the items?

   Should all items be stocked everywhere or should certain items be stocked in only a single location?

3. How much should be ordered when an order is placed?

   Many factors to be considered: demand rate, cost of holding inventory, fixed cost of ordering, ..

4. When should an order be placed?

Economic Order Quantity (EOQ)

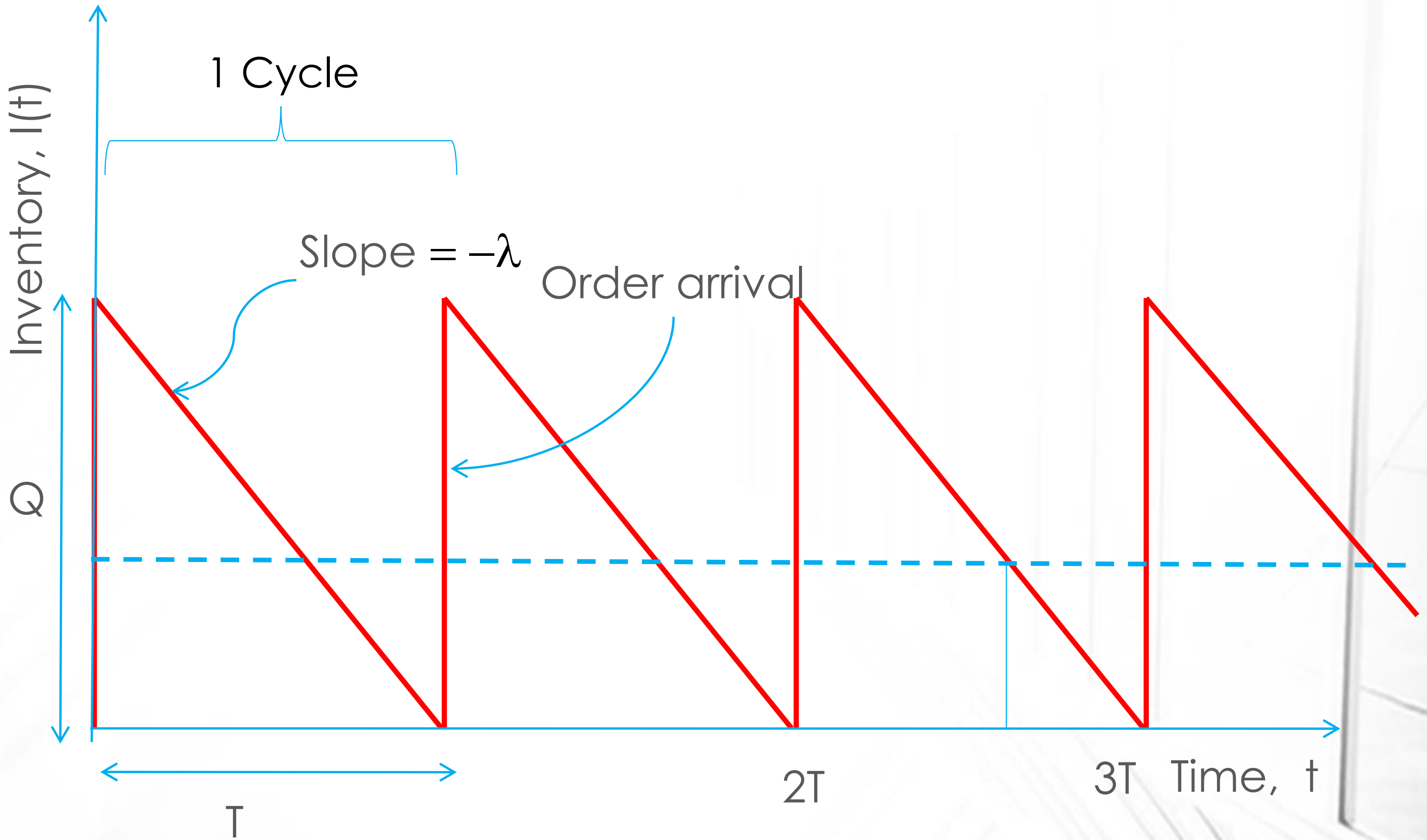
- The EOQ model is the simplest and most fundamental of all inventory models.
- Simplicity and restrictive modelling assumptions usually go together; the EOQ model is not an exception.
- Environment: single-stage system with a single item to stock that has a continuously constant and known demand rate.
- But the model produces good results in many situations and has been effectively employed in automotive, pharmaceutical, and retail sectors of the economy for many years.

EOQ Model

Notations and assumptions underlying the model

- λ – demand rate (units/year) –
  - demand arrives continuously at a constant and known rate
  - Shortages are not permitted(all demand is satisfied on time).
- c – unit order cost ($/unit)
  - not counting setup or inventory cost
- K – fixed or setup cost to place an order ($)
  - When order is placed, it arrives instantly. Order lead time is zero
- Q – Unknown size of the order or lot size
- h – holding cost ($/unit/unit time) (=i*c where i = interest rate)

EOQ Model



1 Cycle
Inventory, I(t)
Slope = $-\lambda$
Order arrival
Q
T
2T
3T
Time, t

EOQ Model

- Our goal is to minimize the average (averaged over time) total cost/unit time by determining the ideal order quantity, Q.

- The time between the placing of two successive orders is the cycle or the reorder interval, T.

→ $T = Q / \lambda$.

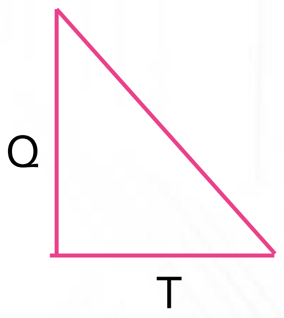
EOQ Model

Cost function

Cost function per unit time (usually a year) to be minimized:

- Purchasing cost for one year = **c*λ**
- Number of orders placed (number of cycles) per year= λ / **Q**
- Annual fixed cost of placing orders= K*(λ /Q)
- Average inventory per cycle : (area of a triangle) / (cycle length).

$$= \frac{\frac{1}{2} Q T}{T} = \frac{Q}{2}$$



- annual cost of holding inventory: h*Q/2

EOQ Model

Cost function

$$G(Q) = \frac{K\lambda}{Q} + \lambda c + \frac{hQ}{2}$$

Annual fixed cost

Annual purchasing cost

Annual holding cost

EOQ Model

Cost function

Min $G(Q) = \frac{K\lambda}{Q} + \lambda c + \frac{hQ}{2}$ where Q>0

$G''(Q) = \frac{d^2G}{dQ^2} = \frac{2K\lambda}{Q^3} > 0 \; for \; Q > 0$

Since G''(Q) >0, it follows that G(Q) is a convex function of Q.

The optimal value of Q occurs where G'(Q) =0.

EOQ Model

Cost function

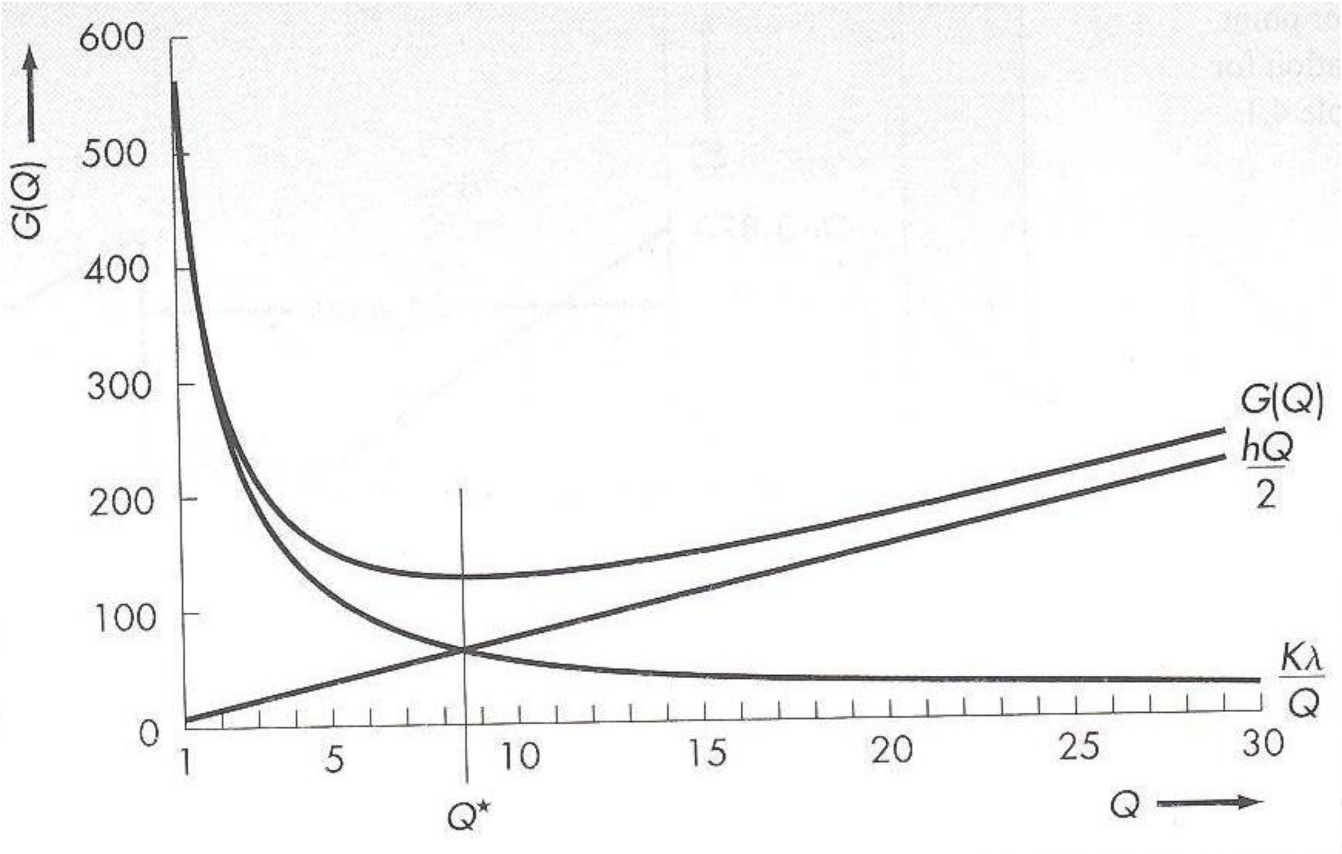
$$\frac{dG}{dQ} = -\frac{K\lambda}{Q^2} + \frac{h}{2} = 0$$

Hence;

$$\boxed{Q = \sqrt{\frac{2K\lambda}{h}} \text{ optimal order quantity !}}$$

EOQ Model

The average annual cost function, G(Q)



Lets say

- λ= 3000 units/year
- c=0,005 $/unit
- K = 0,001 $/order
- h = 6$/unit/year
- $Q = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{2*0{,}001*3000}{6}} = 1$

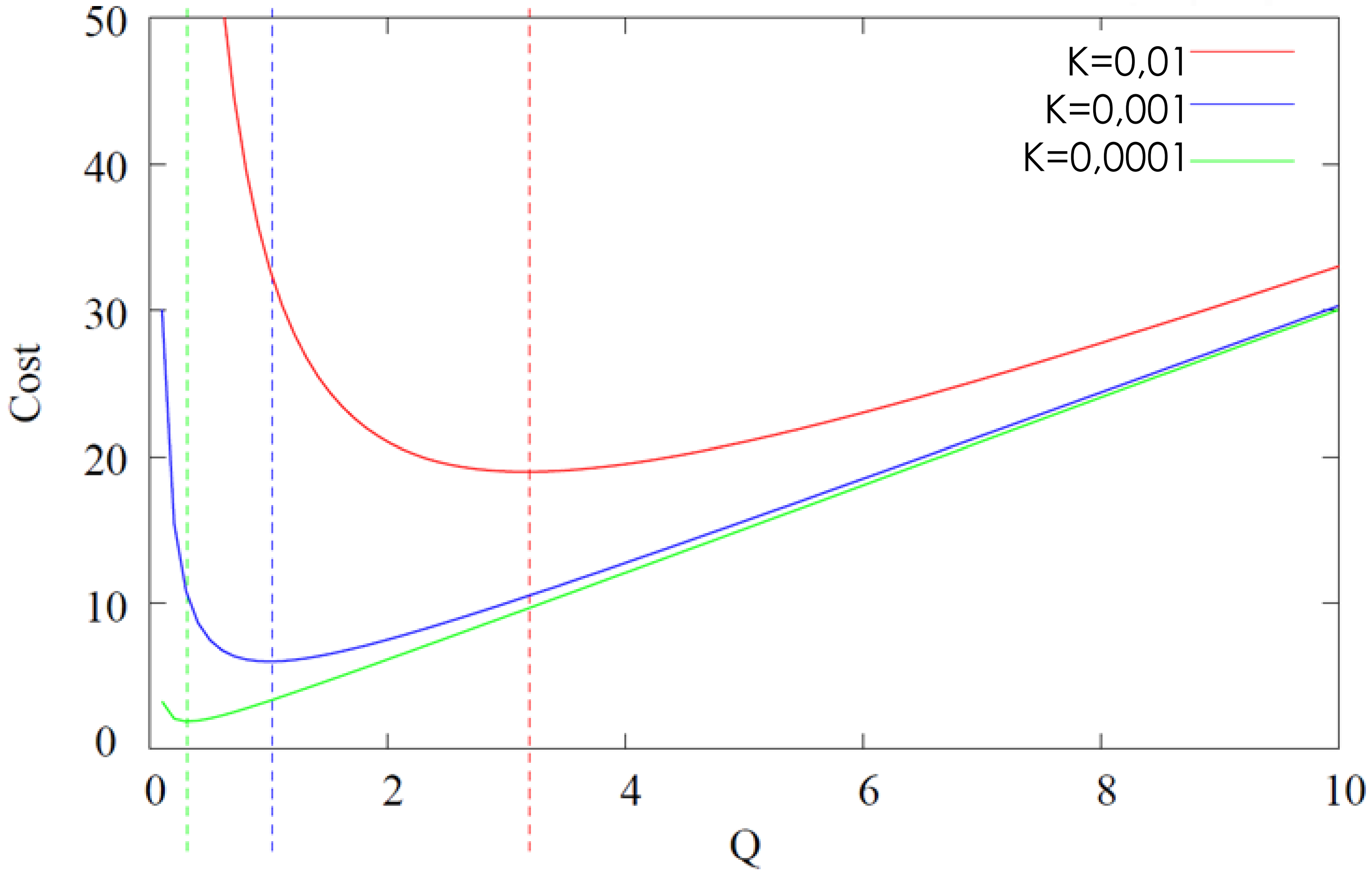
Cost $G(Q) = \frac{K\lambda}{Q} + \lambda c + \frac{hQ}{2} = \frac{0{,}001*3000}{1} + 3000 * 0{,}005 + \frac{6*1}{2}$=21$

Number of orders per year = λ/Q = 3000 orders. Model proposes to have a seperate order for each unit (Q=λ)

WHY?

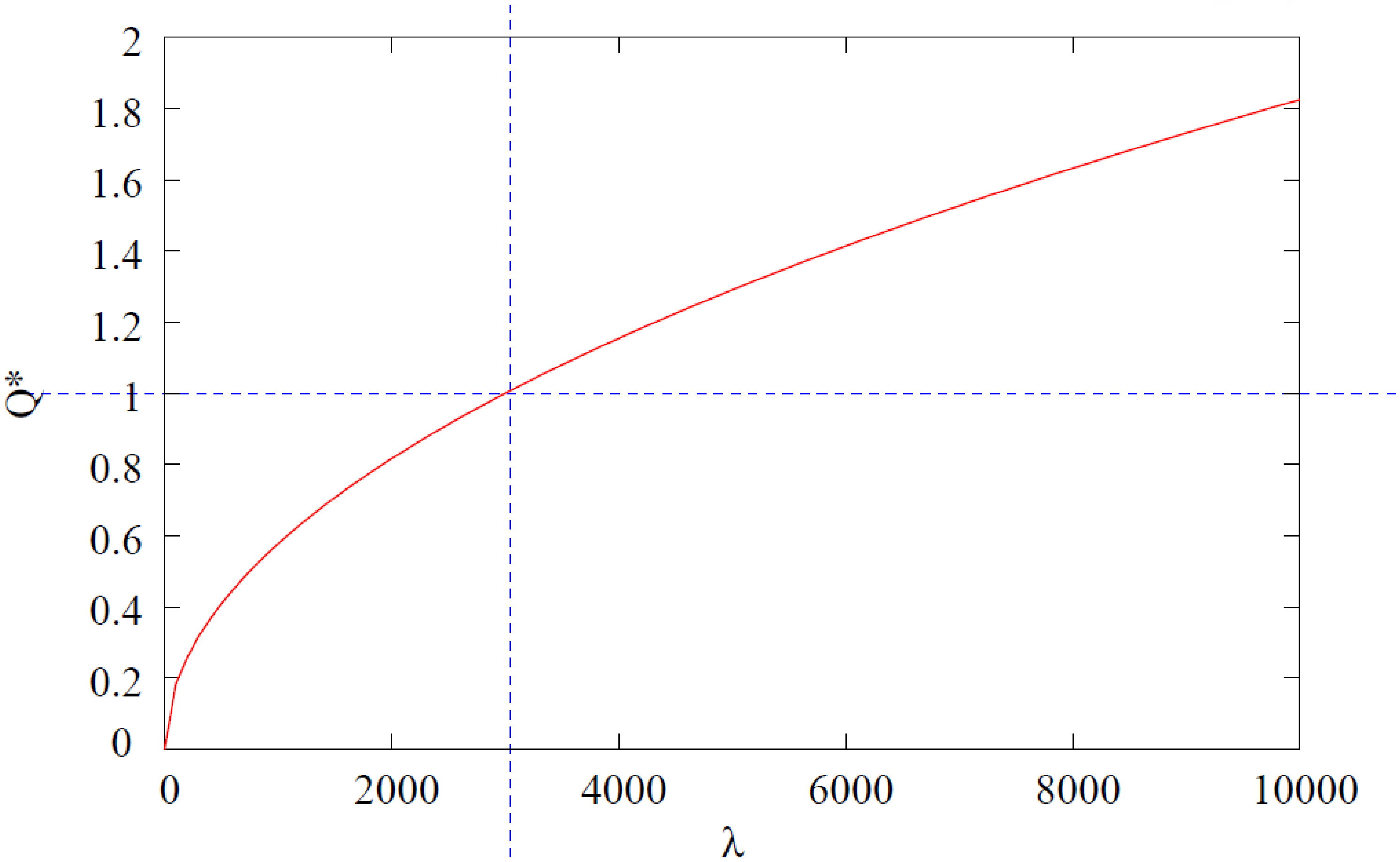
EOQ Model

Effect of order cost on optimal order quantity!



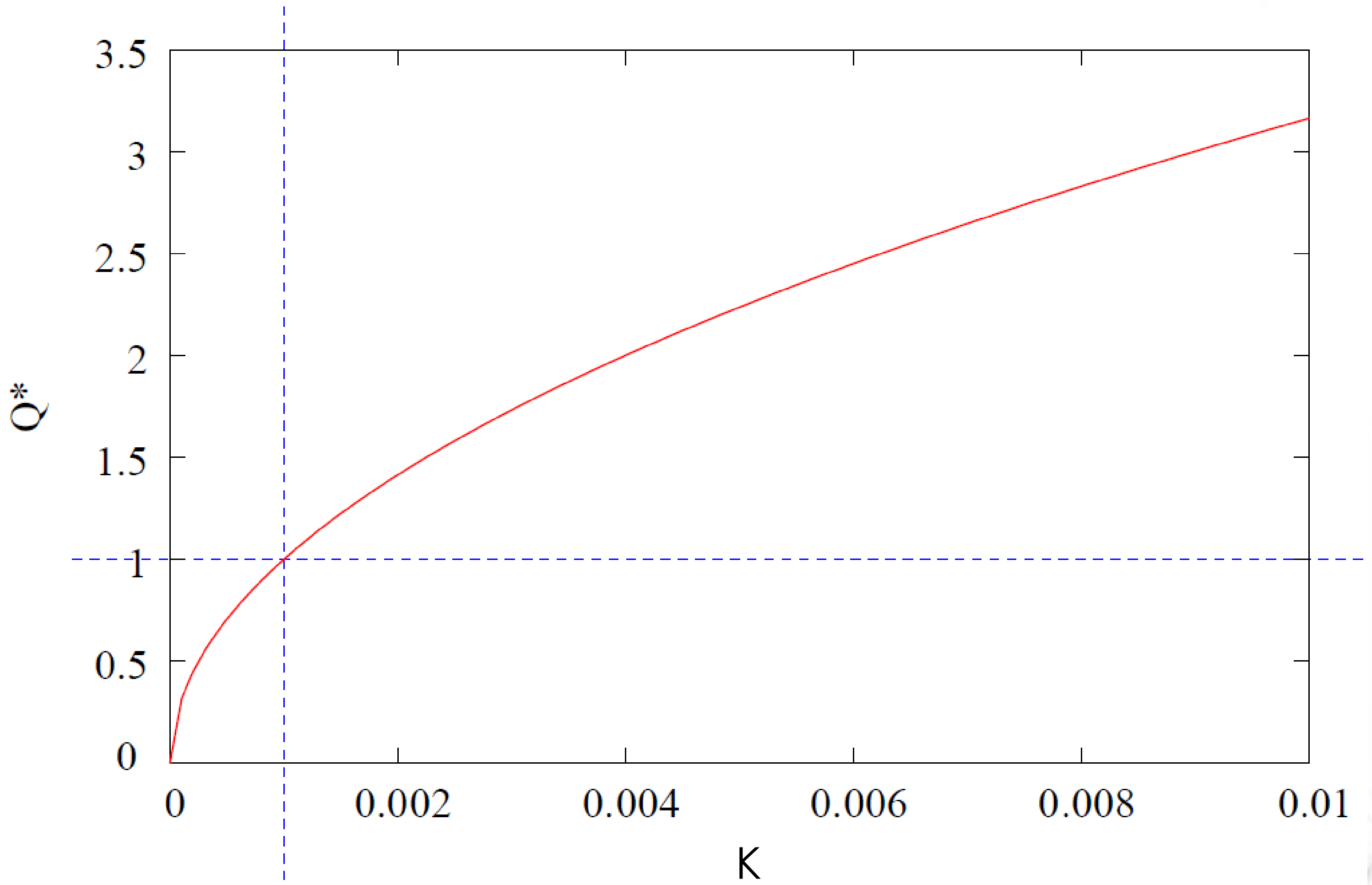
EOQ Model

Relation between Q and λ



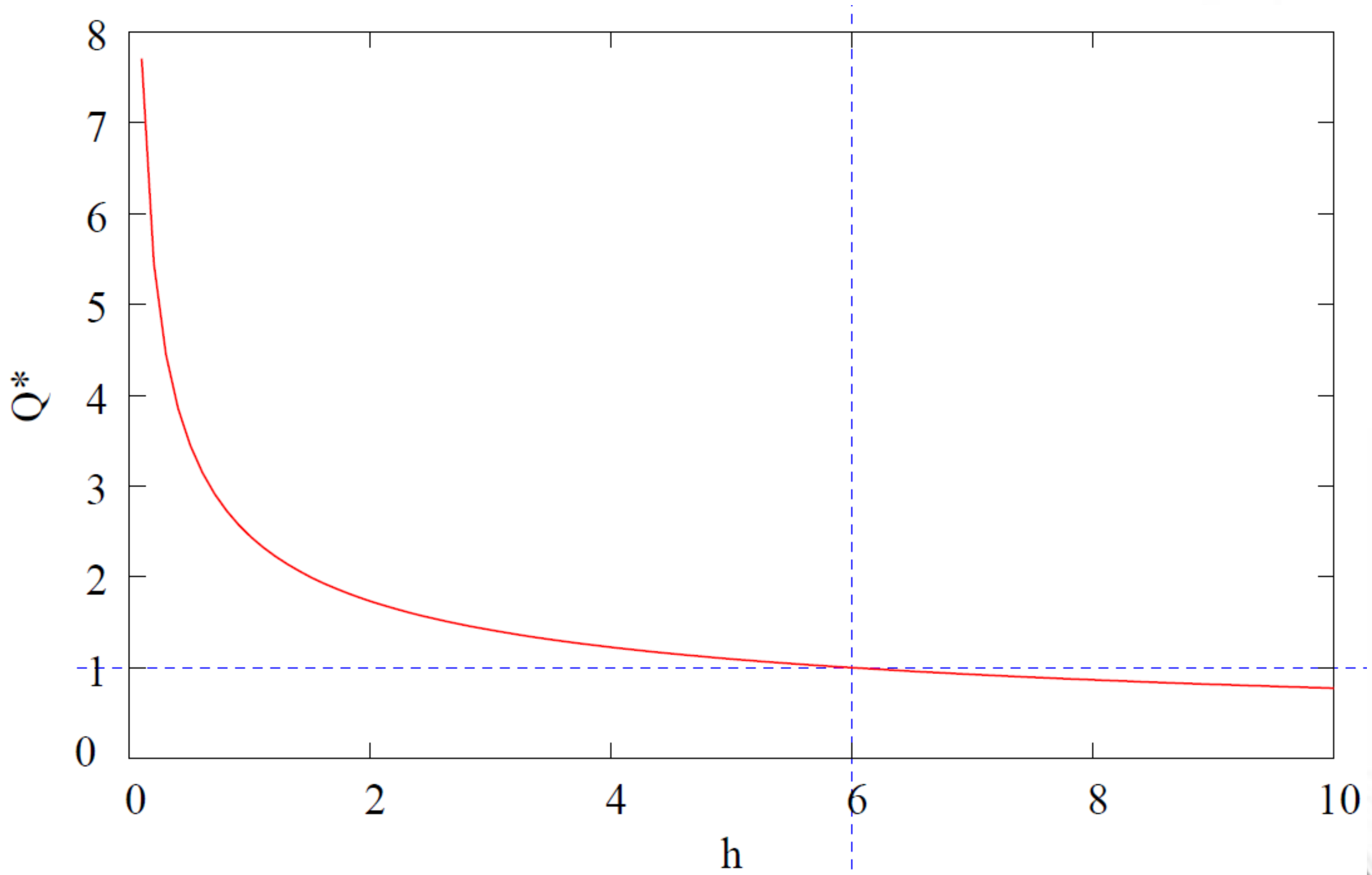
EOQ Model

Relation of Q and K



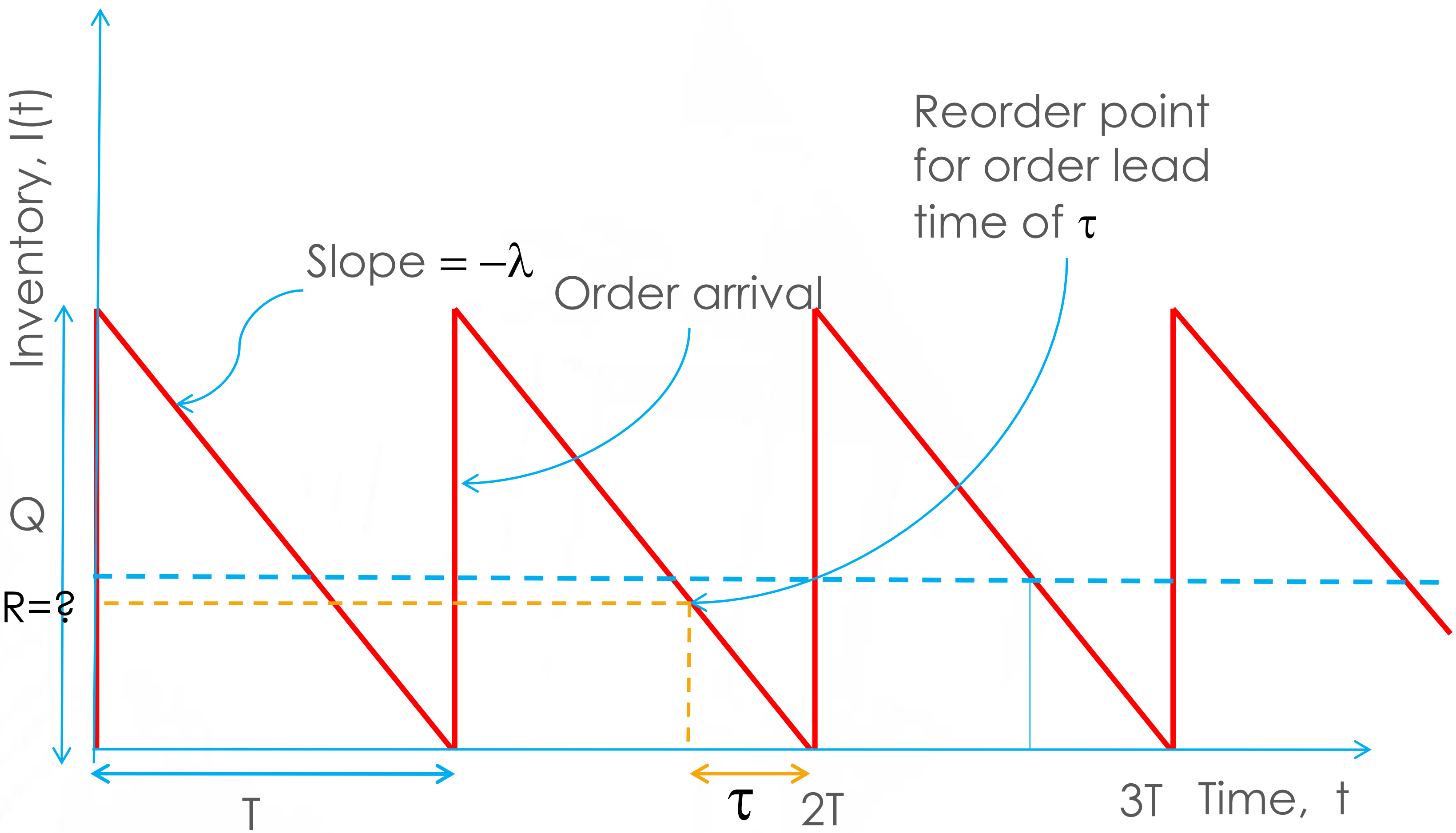
EOQ Model

Relation of Q and h



Nonzero order lead time

If order lead time constraint is relaxed, how can we deal with it?



Nonzero order lead time

- New parameters;
  R - Reorder point (units of item)
  τ – order lead time
- $R = \lambda$ (units/unit time)$^*\tau$ (unit time)
- R represents the reorder point. When the inventory level drops to R, order should be given for the next cycle.
- What if $\tau > T$?
  - Form the ratio τ/T.
  - Consider only the fractional remainder of the ratio, m:

$R^* = \lambda^*(m^*T)$

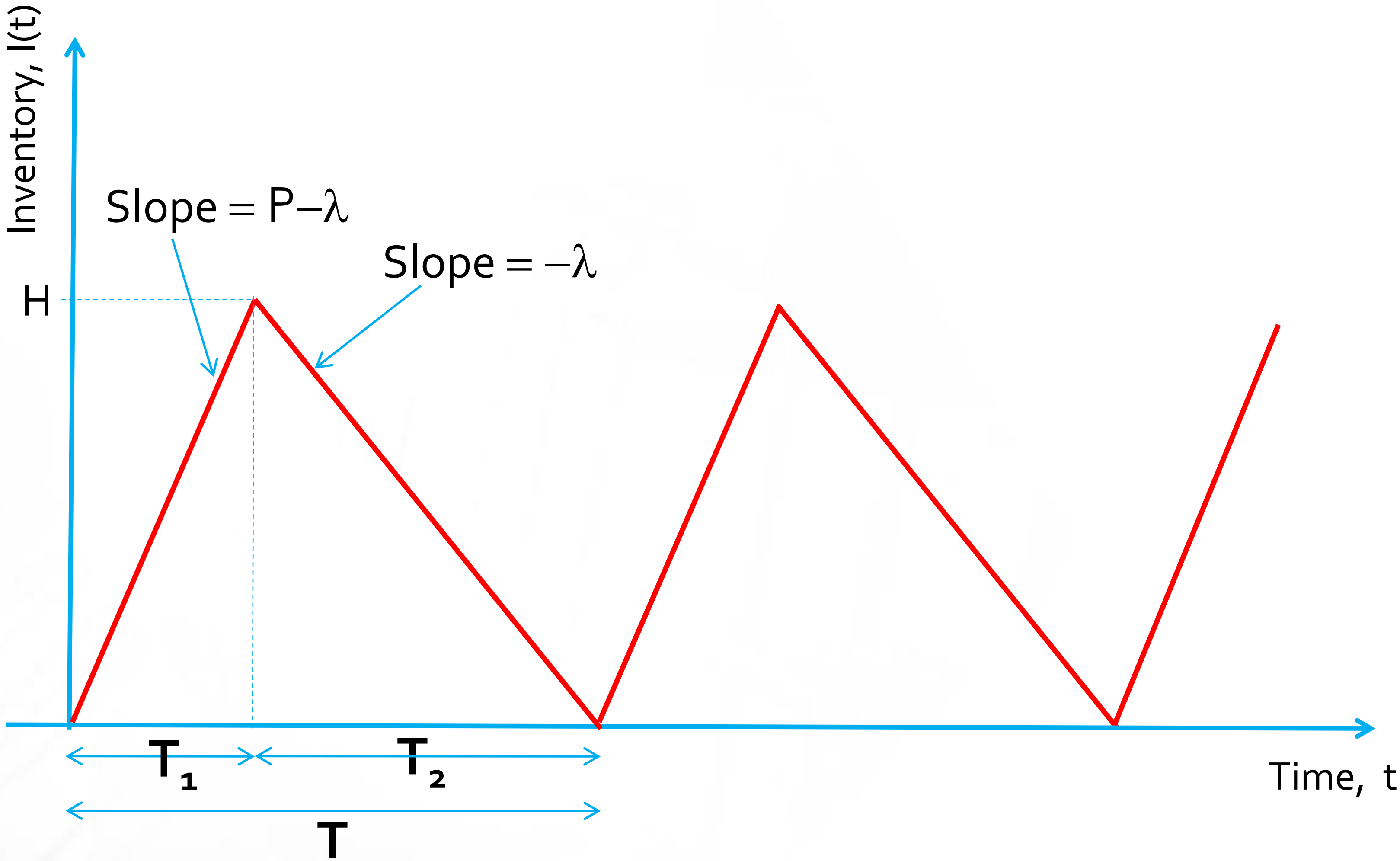
Economic Production Quantity (EPQ)

In EOQ model, it is assumed that orders arrive as a complete lot from outside suppliers. What if we want to produce them internally with a constant rate of production and still benefit EOQ formula?

New parameter;

- P - production rate (units per year), where $P > \lambda$
- All the other assumptions of EOQ are valid.

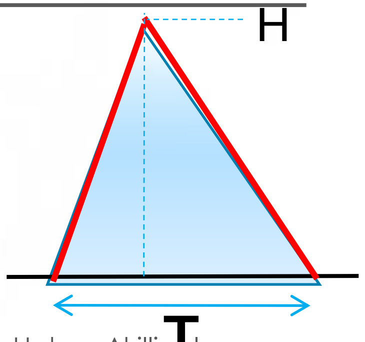
EPQ Model



EPQ Model

- Each order cycle is T: time between successive production startups. T is comprised of two sub-cycles: $T=T_1+T_2$, where T_1 is production time or uptime and T_2 is downtime.
- The number of items produced (= consumed) each cycle is the lot size, $Q= \lambda * T = P*T_1$. → $T_1=Q/P$.
- The maximum level of on-hand inventory is not Q but H (H < Q). $H/T_1 = P-\lambda$ → $H=Q(1 - \lambda/P)$.
- The average inventory per unit time: area of the triangle/T.

$(H*T / 2) / T= H / 2.$



EPQ Model

- The average annual cost function, Z(Q):

$$Z(Q) = c\lambda + \frac{K}{T} + h\frac{H}{2} = c\lambda + \frac{K\lambda}{Q} + \frac{hQ}{2}\left(1 - \lambda/P\right)$$

- *EPQ*=Q* is found by dZ(Q)/dQ=0:

$$EPQ = Q^* = \sqrt{\frac{2K\lambda}{h\left(1-\lambda/P\right)}}$$

- Defining $h' = h(1 - \lambda / P)$

$$EPQ = Q^* = \sqrt{\frac{2K\lambda}{h'}}$$

Quantity discount models

By now, unit cost of product is assumed to be constant,c, being independent from the amount of purchase.

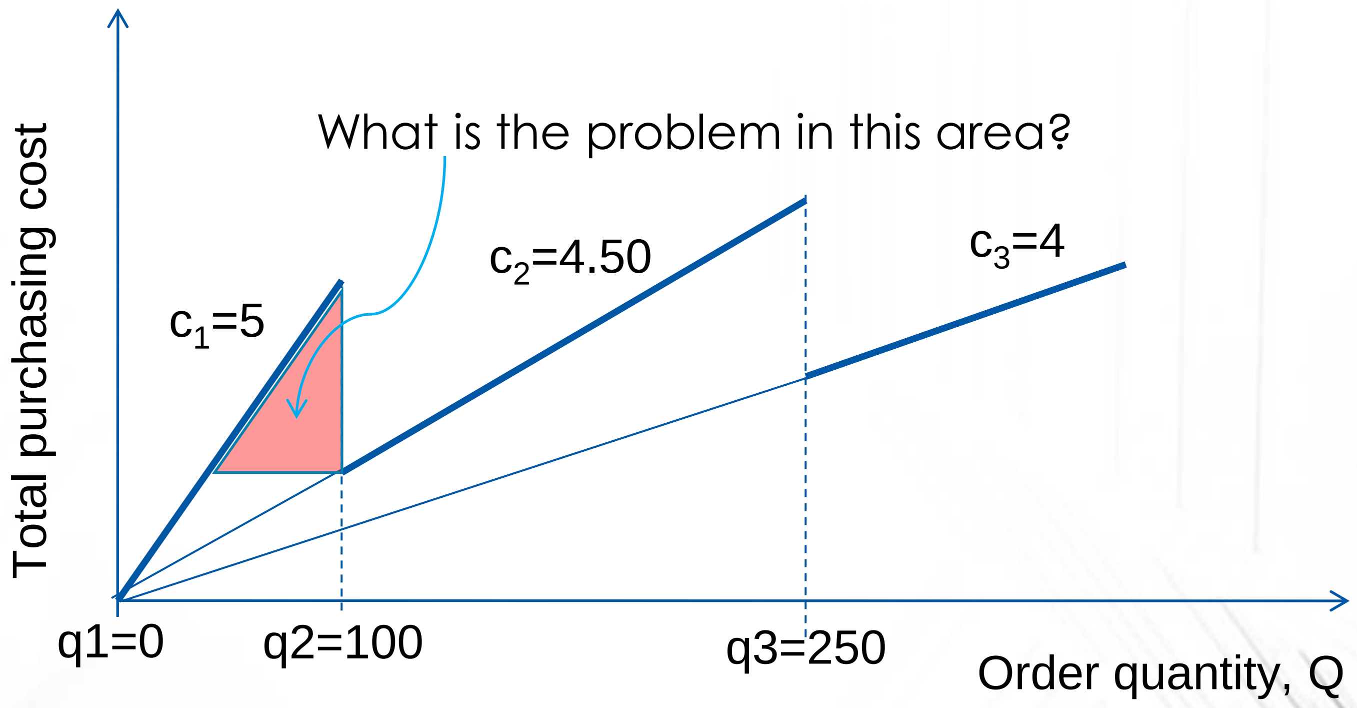
- Lets change it;
  - Unit purchasing cost decreases with the order quantity Q.
  - All the other assumptions of EOQ remain unchanged.
- Two types of quantity discounts are common:
  - all units discounts and incremental quantity discounts.
  - An example of all units discount:

Quantity purchased	Per-unit price
0-100	$5
101-250	$4.50
251 and higher	$4

Quantity discount models in EOQ

All unit discount

- Total purchasing cost for <u>all units discount</u> example:



Quantity discount models in EOQ

All units discount

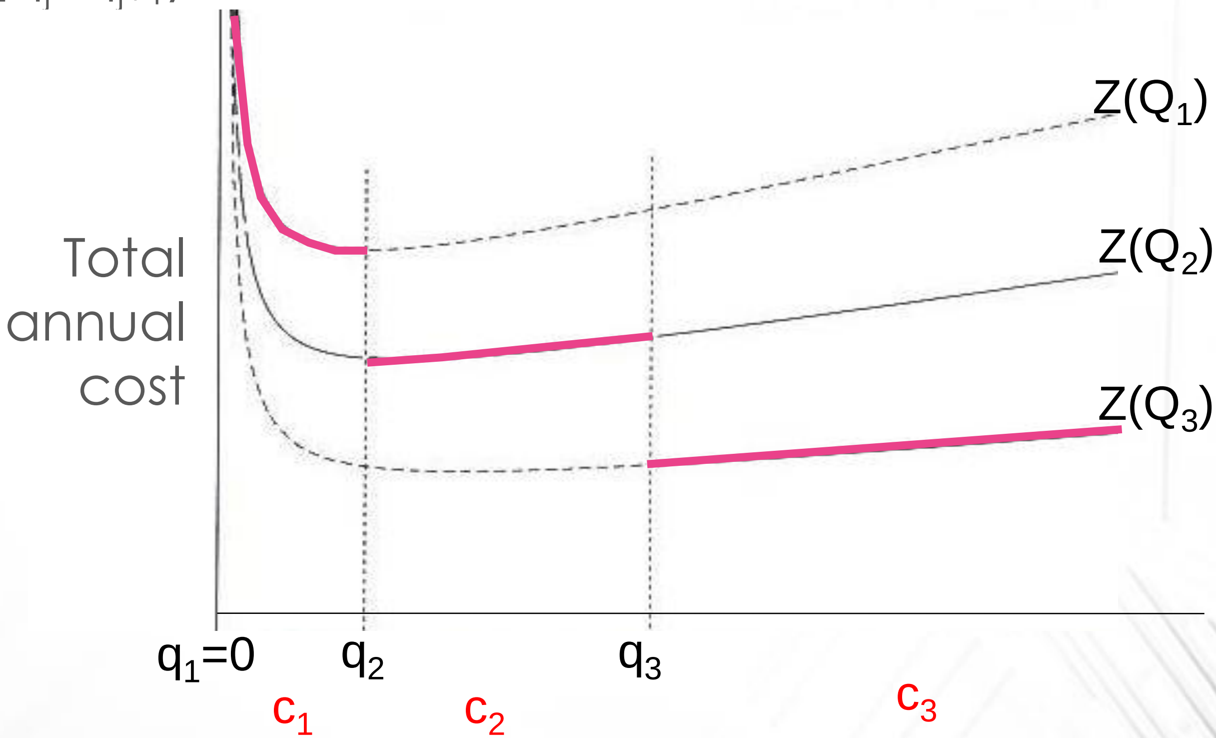
- A more common discount contract.
- One or more price breakpoints defining changes in the unit cost. Let m be the number of discount possibilities. Let $q_1=0, q_2, q_3, .., q_m$ be the order quantities at which the purchasing cost changes.
- Unit purchasing cost in the range $[q_j, q_{j+1})$ is c_j.
- The average annual cost function:

$$Z_j(Q) = \frac{K\lambda}{Q} + c_j\lambda + \frac{Ic_jQ}{2}, \quad q_j \le Q \le q_{j+1}$$

Quantity discount models in EOQ

All units discount

- A family of cost functions indexed by j. The j^{th} cost function is defined for only those values of Q in $[q_j, q_{j+1})$.



Quantity discount models in EOQ

All units discount

Two observations follow:

- The average annual cost function is not continuous. It is segmented such that each segment is defined over a discount interval $[q_j, q_{j+1})$. → makes solving the problem harder.

- The cost curve at the top corresponds to the highest per-unit purchasing cost c_1. The lowest curve corresponds to the lowest per-unit purchasing cost c_m. The curves do not cross each other.

Quantity discount models in EOQ

All units discount

Algorithm to determine Q^*

- Step 1: Set j=m. Compute the EOQ for the *m*th cost curve, denoted by Q_m^*:
- Step 2: Is $Q_m^* \geq q_m$? If yes, Q_m^* is the optimal order quantity and we are done. If not, the minimum cost occurs at $Q= q_m$ for this segment, due to the convexity and non-crossing properties of the cost functions. Compute the cost corresponding to $Q=q_m$.

Quantity discount models in EOQ

All units discount

Let this cost be Z_{min} and $Q_{min} = q_m$ and go to Step 3.

- <u>Step 3</u>: Set j=j-1. Compute the EOQ for the j^{th} cost curve:
- <u>Step 4</u>: Is Q_j^* in $[q_j, q_{j+1})$? If yes, compute $Z(Q_j^*)$ and compare with Z_{min}. If $Z(Q_j^*) < Z_{min}$, Q_j^* is the optimal order quantity; else Q_{min} is the optimal order quantity. In either case we are done.

Quantity discount models in EOQ

All units discount

Step 4 (cont'd)

Otherwise, if Q_j^* is not in $[q_j, q_{j+1})$, then the minimum cost for the j^{th} curve occurs at $Q = q_j$, due to the convexity and non-crossing properties of the cost functions.

Compute the cost, $Z(q_j)$.

If $Z(q_j) < Z_{min}$, then set $Q_{min}=q_j$ and $Z_{min}=Z(q_j)$.

If $j \geq 2$, go to Step 3; otherwise stop.

Quantity discount models in EOQ

All units discount - Example

Example: An office supplies store sees a constant demand rate of 10 boxes of pencil per week. Each box costs \$5. If the fixed cost of placing an order is \$10 and the holding cost rate, i, is 0.20 per year, determine the optimal order quantity using the EOQ model. Assume 52 weeks per year.

Quantity discount models in EOQ

All units discount - Example

- $\lambda = 10*52 = 520$ units/year
- c = $5
- K = $10
- Yearly interest rate = 20%
  - h = i*c = 0,2*5 = $1
- $Q = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{2*10*520}{1}} = 102$

Quantity discount models in EOQ

All units discount - Example

The retailer gets an all units discount of 5% per box of pencils if he purchases at least 110 boxes in a single order. The deal becomes better if the retailer purchases at least 150 boxes in which he gets a 10% discount.

Should the retailer change the order quantity?

Quantity discount models in EOQ

- **All units discount** - Example

SOLUTION:

m=3. c_1=5, c_2=5(1-0.05)=4.75, c_3= 5(1-0.1)=4.50.

q_1=0, q_2=110, q_3=150.

j=3.

Step 1: compute $$Q_3^* = \sqrt{\frac{2(520)(10)}{(0.20)(4.50)}} = 107.5$$

Step 2: Since Q_3^* is not greater than or equal to q_3=150, the minimum cost occurs at Q=150. Its cost Z_{min} is:

$$Z_{min} = (4.50)(520) + \frac{(10)(520)}{150} + \frac{(0.2)(4.50)(150)}{2} = 2442.17$$

Quantity discount models in EOQ

- **All units discount** - Example

Step 3: Set j=2. Compute

$$Q_2^* = \sqrt{\frac{2(520)(10)}{(0.20)(4.75)}} = 104.63$$

Step 3: Q_2^* is not feasible; it is not in [110,150). The minimum feasible cost occurs at Q= q_2=110:

$$Z(q_2) = (4.75)(520) + \frac{(10)(520)}{110} + \frac{(0.2)(4.75)(110)}{2} = 2569.52$$

Quantity discount models in EOQ

All units discount - Example

This cost is higher than Z_{min} and so Z_{min} remains unchanged.

Step 3: j=1. Q_1^*=101.98.

Step 4: Q_1^* is feasible; it is in [0,110). The minimum feasible cost occurs at Q= q_1=101.98

The cost is 2701.98 > Z_{min}.

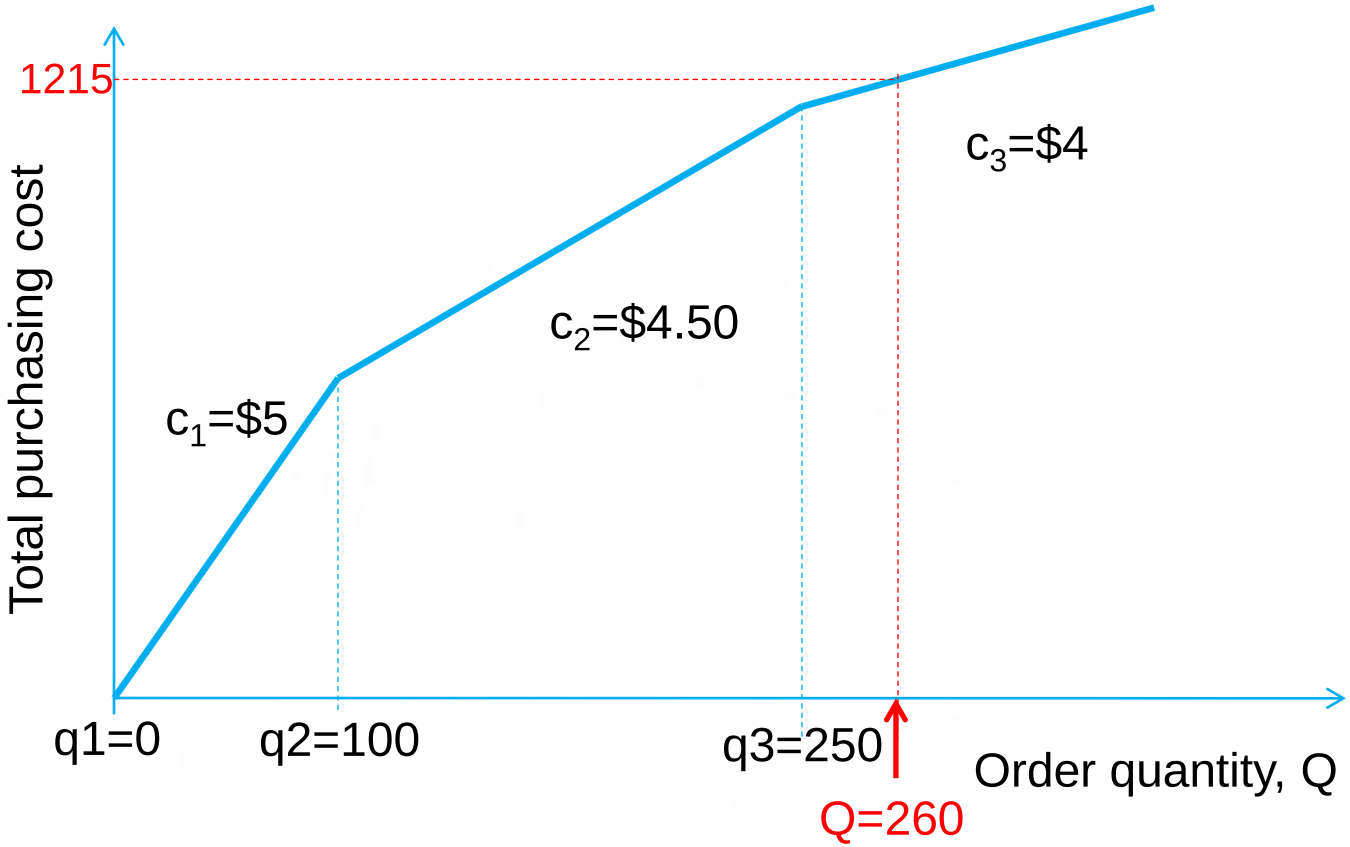
→The optimal solution is to order 150 units with a resulting cost of $2442.17.

→***Observe that the algorithm stops as soon as a discount is found for which Q_j^* is feasible***.

→So change the Q from 102 to 150 !

Quantity discount models in EOQ

Incremental quantity discount



For Q=260 units, total purch. cost=(5)(100)+(4.5)(150)+(4)(10)=$1215

Quantity discount models in EOQ

Incremental quantity discount

- Differs from the all units discount contracts.
- As Q increases, the unit purchasing cost, c_j, declines incrementally on additional units purchased.
- Let m be the number of discount levels. Let $q_1=0$, q_2, q_3, .. , q_m be the order quantities at which the unit purchasing cost changes.
- Unit purchasing cost in the range $[q_j, q_{j+1}]$ is c_j.

Quantity discount models in EOQ

- **Incremental quantity discount**

- If Q units in the j^{th} discount interval $[q_j, q_{j+1}]$ are ordered, the purchasing cost is:

$$C(Q) = \underbrace{c_1(q_2 - q_1) + c_2(q_3 - q_2) + \ldots + c_{j-1}(q_j - q_{j-1})}_{R_j,\ j\geq 2} + c_j(Q - q_j).$$

$$C(Q) = R_j + c_j(Q - q_j).$$

- The average unit purchasing cost for Q units is $C(Q)/Q$.

Quantity discount models in EOQ

- **Incremental quantity discount**

$$\frac{C(Q)}{Q} = \frac{R_j}{Q} + c_j - c_j \frac{q_j}{Q}.$$

- The average annual cost function, Z(Q):

$$Z(Q) = \frac{C(Q)}{Q}\lambda + K\frac{\lambda}{Q} + \frac{i\frac{C(Q)}{Q}Q}{2}$$

$$= \left(\frac{R_j}{Q} + c_j - c_j \frac{q_j}{Q}\right)\lambda + K\frac{\lambda}{Q} + \frac{i(R_j + c_j(Q - q_j))}{2}$$

Quantity discount models in EOQ

Incremental quantity discount

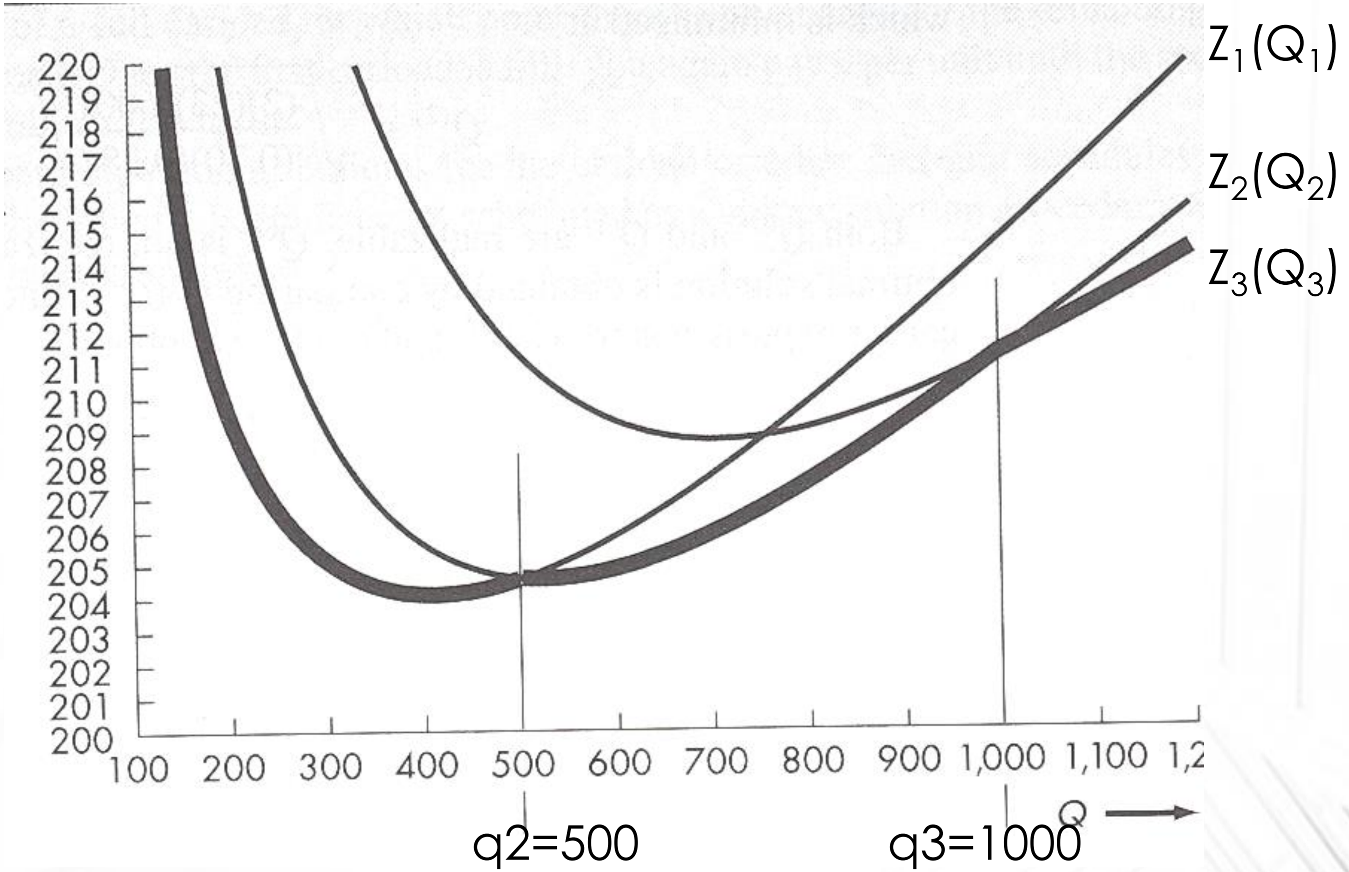
- Rearranging the terms in Z(Q) and naming it as $Z_j(Q)$:

$$Z_j(Q) = c_j\lambda + (R_j - c_jq_j + K)\frac{\lambda}{Q} + \frac{ic_jQ}{2} + \frac{i(R_j - c_j q_j)}{2} \quad for \;\; q_j \leq Q \leq q_{j+1}$$

- Then we have a family of curves, $Z_j(Q)$ for each j: each curve valid for a given interval is convex and differentiable. The curves $Z_j(Q)$ and $Z_{j+1}(Q)$ cross at q_{j+1}.

Quantity discount models in EOQ

Incremental quantity discount



Quantity discount models in EOQ

Incremental quantity discount

Algorithm to determine Q^*

- Step 1: Compute the order quantity that minimizes $Z_j(Q)$ for each j, which is denoted by Q_j^* and obtained by setting $d\,Z_j(Q)\,/dQ=0$

$$\frac{dZ_j(Q)}{dQ} = -(R_j - c_j q_j + K)\frac{\lambda}{Q^2} + \frac{ic_j}{2}$$

$$\Rightarrow Q_j^* = \sqrt{\frac{2(R_j - c_j q_j + K)\lambda}{i c_j}}$$

Quantity discount models in EOQ

Incremental quantity discount

- This step gives us a total of *m* possible order quantities.
- Step 2: We check the feasibility of the potential values for Q^*, that is, $q_j \leq Q_j^* \leq q_{j+1}$? Disregard the ones that do not satisfy this inequality.
- Step 3: Calculate the cost $Z_j(Q_j^*)$ for each remaining Q_j^*. The order quantity Q_j^* that gives the least cost is the optimal order quantity.

Quantity discount models in EOQ

Incremental quantity discount - Example

Office supplies again. The retailer is offered an incremental quantity discount.

$c_1=5$, $c_2=4.75$, $c_3=4.50$.

$q_1=0$, $q_2=109$, $q_3=149$.

Solution:

Step 1:
$R_1=0$,
$R_2= c_1(q_2- q_1)=(5)(109-0)=545$,
$R_3= c_1(q_2- q_1)+ c_2(q_3- q_2)= R_2+(4.75)(149-109)=735$.

Compute Q_j^* values.

Quantity discount models in EOQ

Incremental quantity discount - Example

Step 1: cont'd.

$$Q_1^* = \sqrt{\frac{2(R_1 - c_1 q_1 + K)\lambda}{i c_1}} = \sqrt{\frac{2(0 - 0 + 10)(520)}{(0.2)(5)}} = 101.98$$

$$Q_2^* = \sqrt{\frac{2(R_2 - c_2 q_2 + K)\lambda}{i c_2}} = \sqrt{\frac{2(545 - (4.75)(109) + 10)(520)}{(0.2)(4.75)}} = 201.94$$

$$Q_3^* = \sqrt{\frac{2(R_3 - c_3 q_3 + K)\lambda}{i c_3}} = \sqrt{\frac{2(735 - (4.5)(149) + 10)(520)}{(0.2)(4.5)}} = 293.41$$

Quantity discount models in EOQ

- Step 2: We disregard Q_2^* since Q_2^* is not in [110,149]. Q_1^* and Q_3^* are feasible.

- Step 3: We compute the costs for Q_1^* and Q_3^*:

$$Z(Q_1^*) = c_1\lambda + (R_1 - c_1 q_1 + K)\frac{\lambda}{Q_1^*} + \frac{ic_1 Q_1^*}{2} + \frac{i(R_1 - c_1 q_1)}{2}$$

$$= (5)(520) + (0 - 0 + 10)\frac{520}{101.98} + \frac{(0.2)(5)(101.98)}{2} + \frac{(0.2)(0-0)}{2} = 2701.98$$

$$Z(Q_3^*) = c_3\lambda + (R_3 - c_3 q_3 + K)\frac{\lambda}{Q_3^*} + \frac{ic_3 Q_3^*}{2} + \frac{i(R_3 - c_3 q_3)}{2}.$$

$$= (4.5)(520) + (735 - (4.5)(149) + 10)\frac{520}{293.41} + \frac{(0.2)(4.5)(293.41)}{2} + \frac{(0.2)(735 - (4.5)(149))}{2} = 2610.52$$

The optimal solution is $Q^* = 293.41$ with a resulting cost of \$2610.52 > \$2442.17 (all-units discount case).