Logic systems

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Lecture 01

Propositions 1

A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

- Washington, D.C., is the capital of the United States of America.
- Toronto is the capital of Canada.
- 1 + 1 = 2.
- 2 + 2 = 3.

Propositions 2

Some sentences that are not propositions:

- What time is it?
- Read this carefully!
- $x + 1 = 2$.
- $x + y = z$.

Propositions 3

- We use letters to denote **propositional variables** (or **sentential variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s ...
- The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition. Propositions that cannot be expressed in terms of simpler propositions are called **atomic propositions**.

Propositional calculus

- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.
- Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using **logical operators**.

Negation

- Let p be a proposition. The **negation** of p, denoted by ¬p, is the statement "It is not the case that p." The proposition ¬p is read "not p." The truth value of the negation of p, ¬p, is the opposite of the truth value of p.
- Table displays the **truth table** for the negation of a proposition p.

p	$\neg p$
T	F
F	T

Negation - example

- Task: Find the negation of the proposition “Vandana’s smartphone has at least 32 GB of memory” and express this in simple English.
- Solution: The negation is “It is not the case that Vandana’s smartphone has at least 32 GB of memory.” This negation can also be expressed as “Vandana’s smartphone does not have at least 32 GB of memory” or even more simply as “Vandana’s smartphone has less than 32 GB of memory.”

Negation - conclusion

The negation of a proposition can also be considered the result of the operation of the negation operator on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

Conjunction

- Let p and q be propositions. The conjunction of p and q, denoted by p ∧ q, is the proposition "p and q." The conjunction p ∧ q is true when both p and q are true and is false otherwise.
- The truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Conjunction example

- Task: Find the conjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”
- Solution: The conjunction of these propositions, $p \wedge q$, is the proposition “Rebecca’s PC has more than 16 GB free hard disk space, and the processor in Rebecca’s PC runs faster than 1 GHz.” This conjunction can be expressed more simply as “Rebecca’s PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz.” For this conjunction to be true, both conditions given must be true. It is false when one or both of these conditions are false.

Disjunction

- Let p and q be propositions. The disjunction of p and q, denoted by p V q, is the proposition "p or q." The disjunction p V q is false when both p and q are false and is true otherwise.
- The use of the connective or in a disjunction corresponds to one of the two ways the word or is used in English, namely, as an inclusive or. A disjunction is true when at least one of the two propositions is true. That is, p V q is true when both p and q are true or when exactly one of p and q is true.
- The truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive or

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$ (or p XOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example

- Let p and q be the propositions that state “A student can have a salad with dinner” and “A student can have soup with dinner,” respectively. What is $p \oplus q$, the exclusive or of p and q?
- The exclusive or of p and q is the statement that is true when exactly one of p and q is true. That is, $p \oplus q$ is the statement “A student can have soup or salad, but not both, with dinner.” Note that this is often stated as “A student can have soup or a salad with dinner,” without explicitly stating that taking both is not permitted.

Conditional statements

- Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the **hypothesis** (or antecedent or premise) and q is called the **conclusion** (or consequence).
- The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

Implication truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication truth value explanation 1

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is “If I am elected, then I will lower taxes.” If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \rightarrow q$.

Implication truth value explanation 2

Similarly, consider a statement that a professor might make: “If you get 100% on the final, then you will get an A.” If you manage to get 100% on the final, then you would expect to receive an A. If you do not get 100%, you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

Note

- Most programming languages contain statements such as **if** p **then** S, where p is a proposition and S is a program segment (one or more statements to be executed). (Although this looks as if it might be a conditional statement, S is not a proposition, but rather is a set of executable instructions.) When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false.

Converse, contrapositive, and inverse

We can form some new conditional statements starting with a conditional statement $p \rightarrow q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$. The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$. We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

Converse, contrapositive, and inverse

Conditional Statement	$p \rightarrow q$	If p, then q
Converse	$q \rightarrow p$	If q, then p
Inverse	$\sim p \rightarrow \sim q$	If not p, then not q
Contrapositive	$\sim q \rightarrow \sim p$	If not q, then not p

Conditional Contrapositive

p	q	~p	~q	p→q	q→p	~p→~q	~q→~p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Converse Inverse

Example

- Find the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining."
- Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If it is raining, then the home team wins." Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining." The converse is "If the home team wins, then it is raining." The inverse is "If it is not raining, then the home team does not win." Only the contrapositive is equivalent to the original statement.

Logical equivalence

When two compound propositions always have the same truth values, regardless of the truth values of its propositional variables, we call them **equivalent**. Hence, a conditional statement and its contrapositive are equivalent.

Compound propositions

- Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

- $(p \lor q) \land (\neg r)$
- $(p \lor q) \land \neg r$

Operator	*Precedence*
$\neg$	1
$\land$ $\lor$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Logic and Bit Operations

- Computers represent information using bits. A **bit** is a symbol with two possible values, namely 0 (zero) and 1 (one). This meaning of the word bit comes from *b*inary dig*it*, because zeros and ones are the digits used in binary representations of numbers. A variable is called a **Boolean variable** if its value is either true or false.

Truth Value	*Bit*
T	1
F	0

Table for the Bit Operators OR, AND, and XOR

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Translating English Sentences 1

- How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

- We let a, c, and f represent "You can **a**ccess the Internet from campus," "You are a **c**omputer science major," and "You are a **f**reshman," respectively.
- $a \rightarrow (c \lor \neg f)$

Translating English Sentences 2

- How can this English sentence be translated into a logical expression? “You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”
- Let r, f, and o represent “You can **r**ide the roller coaster,” “You are under 4 **f**eet tall,” and “You are **o**lder than 16 years old,” respectively.
- $(f \land \neg o) \rightarrow \neg r$

Translating English Sentences 3 - 1

- As a reward for saving his daughter from pirates, the King has given you the opportunity to win a treasure hidden inside one of three trunks. The two trunks that do not hold the treasure are empty. To win, you must select the correct trunk. Trunks 1 and 2 are each inscribed with the message “This trunk is empty,” and Trunk 3 is inscribed with the message “The treasure is in Trunk 2.” The Queen, who never lies, tells you that only one of these inscriptions is true, while the other two are wrong. Which trunk should you select to win?

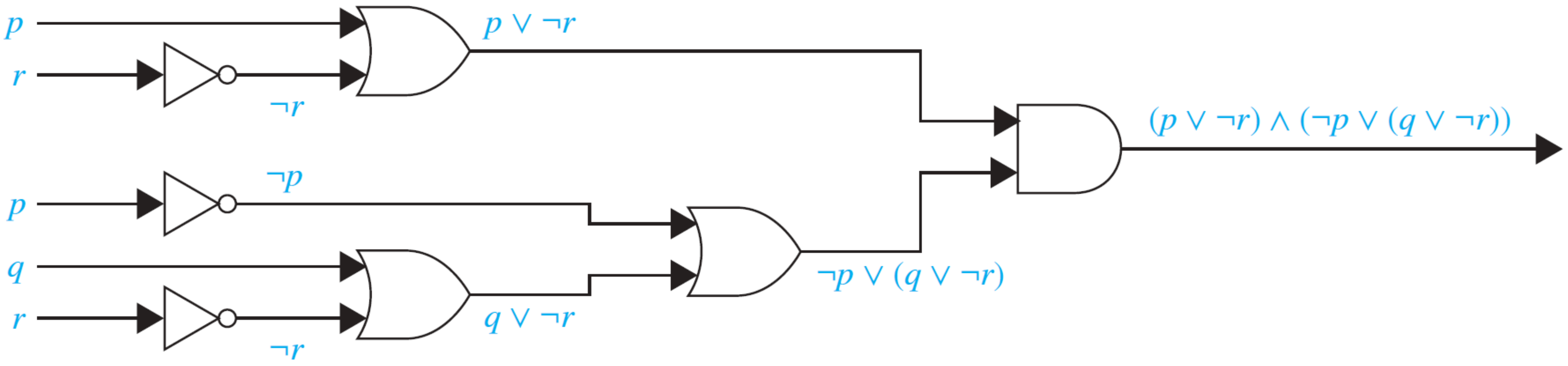
Translating English Sentences 3 - 2

Let p_i be the proposition that the treasure is in Trunk i, for i = 1, 2, 3. To translate into propositional logic the Queen's statement that exactly one of the inscriptions is true, we observe that the inscriptions on Trunk 1, Trunk 2, and Trunk 3, are ¬p1, ¬p2, and p2, respectively. So, her statement can be translated to

$$(\neg p1 \wedge \neg(\neg p2) \wedge \neg p2) \vee (\neg(\neg p1) \wedge \neg p2 \wedge \neg p2) \vee (\neg(\neg p1) \wedge \neg(\neg p2) \wedge p2))$$

Logic circuits

- Determine the logic circuit for: (p∨¬r) ∧ (¬p ∨ (q∨¬r))



Propositional equivalences

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. A compound proposition that is always false is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.
- The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example 1

- Show that ¬(p ∨ q) and ¬p ∧ ¬q are logically equivalent
- We use the truth table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Example 2

- Show that p → q and ¬p ∨ q are logically equivalent.
- We construct the truth table

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 3

- Show that p ∨ (q ∧ r) and (p ∨ q) ∧ (p ∨ r) are logically equivalent. This is the distributive law of disjunction over conjunction.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Modus ponens

- $(p \wedge (p \rightarrow q)) \rightarrow q$

Predicates

- Statements involving variables, such as “x > 3,” “x = y + 3,” “x + y = z,” are often found in mathematical assertions.
- They are called **predicates** (or also open statements)
- We can denote the statement “x is greater than 3” by P(x), where P denotes the predicate “is greater than 3” and x is the variable.
- The statement $P(x)$ is also said to be the value of the **propositional function** P at x.

Example

- Let P(x) denote the statement “x > 3.” What are the truth values of P(4) and P(2)?
- We obtain the statement P(4) by setting x = 4 in the statement “x > 3.” Hence, P(4), which is the statement “4 > 3,” is true. However, P(2), which is the statement “2 > 3,” is false.

Quantifiers

- When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification**, to create a proposition from a propositional function.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

The universal quantifier

- The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain."
- The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier**.
- We read $\forall x P(x)$ as "for all $x P(x)$" or "for every x $P(x)$."

Examples

- Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?
- Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is true.
- Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?
- $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus, $\forall x Q(x)$ is false.

Existential quantifier

- The existential quantification of P(x) is the proposition "There exists an element x in the domain such that P(x)."
- We use the notation $\exists x P(x)$ for the existential quantification of P(x). Here $\exists$ is called the **existential quantifier**.

Examples

- Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?
- Because "$x > 3$" is sometimes true (for instance, when $x = 4$), the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.
- Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?
- Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false.

Negating Quantified Expressions

- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$

Example

- What are the negations of the statements “There is an honest politician” and “All Americans eat cheeseburgers”?
- Let $H(x)$ denote “x is honest.” Then the statement “There is an honest politician” is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg \exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as “Every politician is dishonest.” (Note: In English, the statement “All politicians are not honest” is ambiguous. In common usage, this statement often means “Not all politicians are honest.” Consequently, we do not use this statement to express this negation.) Let $C(x)$ denote “x eats cheeseburgers.” Then the statement “All Americans eat cheeseburgers” is represented by $\forall x C(x)$, where the domain consists of all Americans. The negation of this statement is $\neg \forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including “Some American does not eat cheeseburgers” and “There is an American who does not eat cheeseburgers.”

Translating from English into Logical Expressions - 1

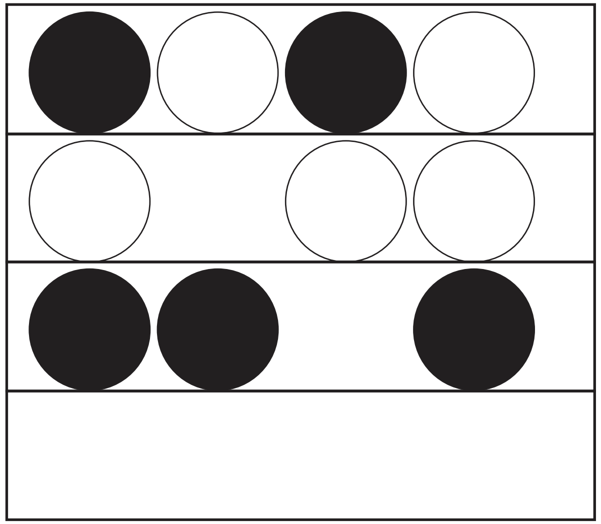
- Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.
- First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain: "For every student in this class, that student has studied calculus." Next, we introduce a variable x so that our statement becomes "For every student x in this class, x has studied calculus." Continuing, we introduce C(x), which is the statement "x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

Translating from English into Logical Expressions - 2

- If we change the domain to consist of all people, we will need to express our statement as "For every person x, if person x is a student in this class, then x has studied calculus."
- If S(x) represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$. [Caution! Our statement cannot be expressed as $\forall x(S(x) \land C(x))$ because this statement says that all people are students in this class and have studied calculus!]

Dilemma

$\forall x P(x)$ or $\exists x \neg P(x)$



Nested quantifiers

- $\forall x \forall y(x + y = y + x)$
- $\forall x \exists y(x + y = 0)$
- $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

Order of quantifiers

- $\forall x \exists y \; (y > x)$
- $\exists y \forall x \; (y > x)$

Example

- Limit in mathematical terms

Sets

A set is an unordered collection of distinct objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

- Roster method: $V = \{a, e, i, o, u\}$
- Set builder notation: $O = \{x \mid x$ is an odd positive integer less than $10\}$

Equality of sets

- Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.
- The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

Empty set

- There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$.

Subsets

- The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)
- We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ is true.

Cartesian Products

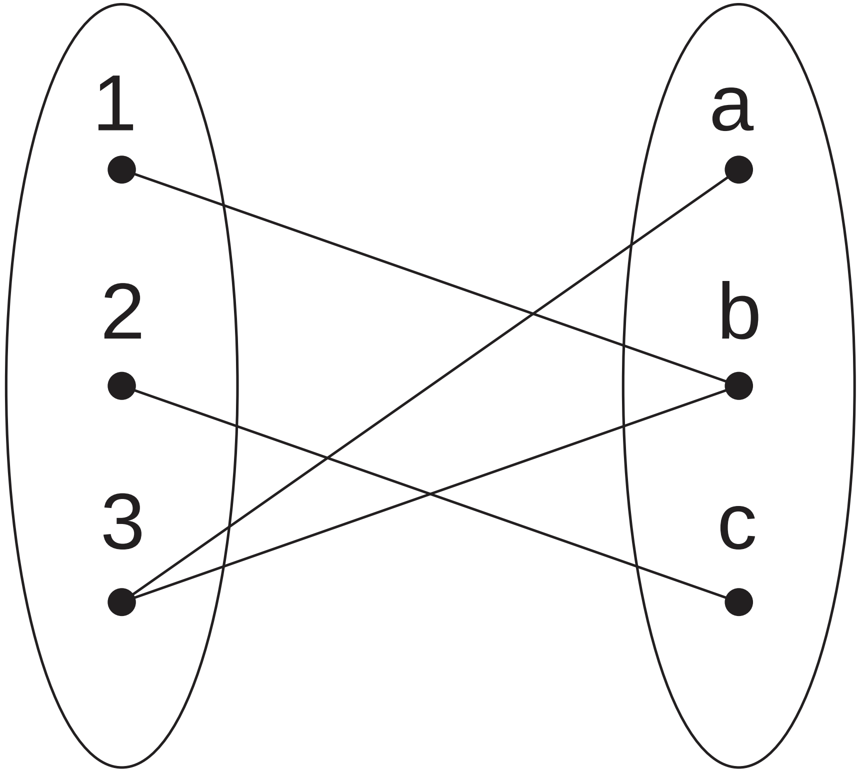
- Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$.

Example

- What is the Cartesian product of A = {1, 2} and B = {a, b, c}?
- The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

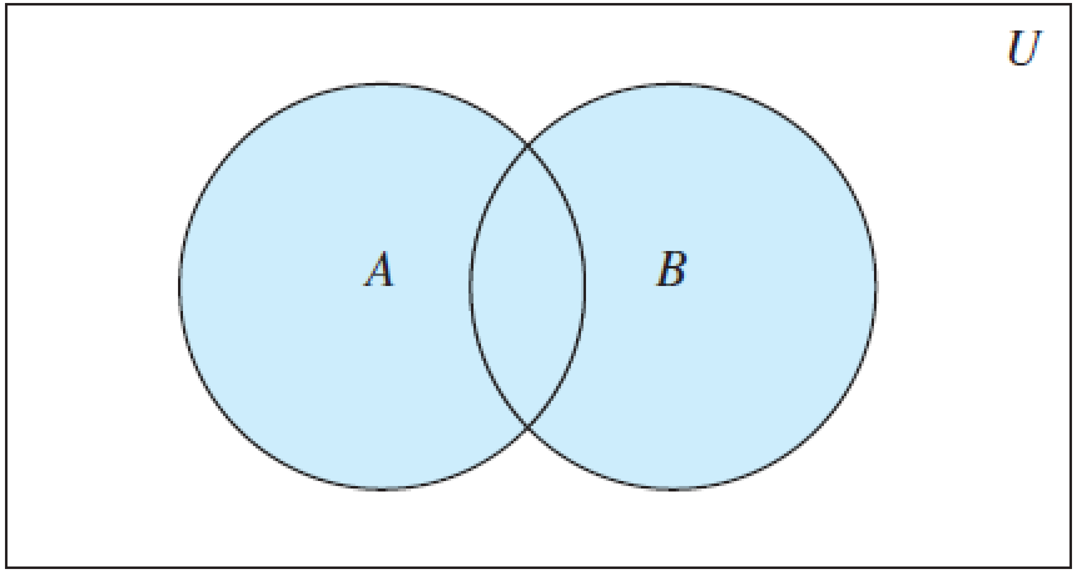
Relations

- Relations are subsets of cartesian products



Union of sets

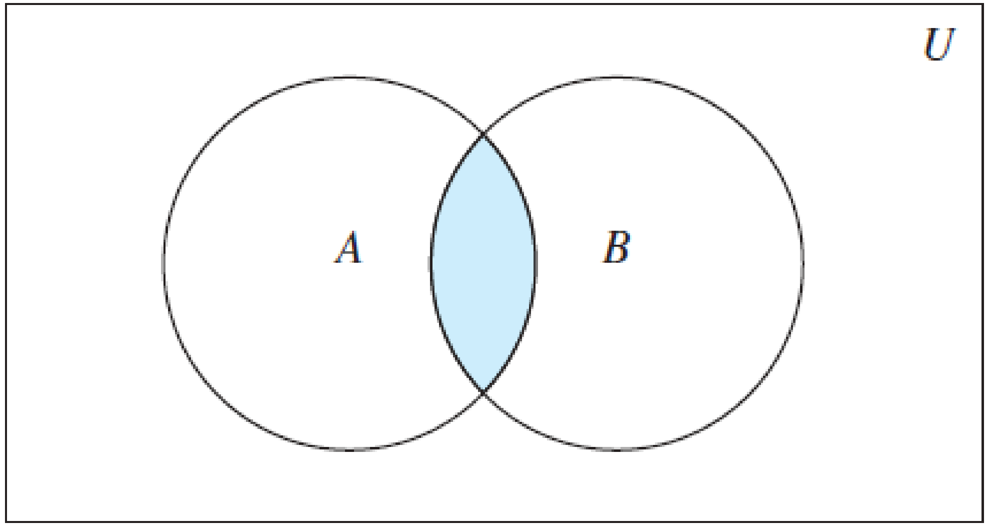
- Let A and B be sets. The **union** of the sets A and B, denoted by A ∪ B, is the set that contains those elements that are either in A or in B, or in both.
- $A \cup B = \{x \mid x \in A \lor x \in B\}$.



$A \cup B$ is shaded.

Intersection of sets

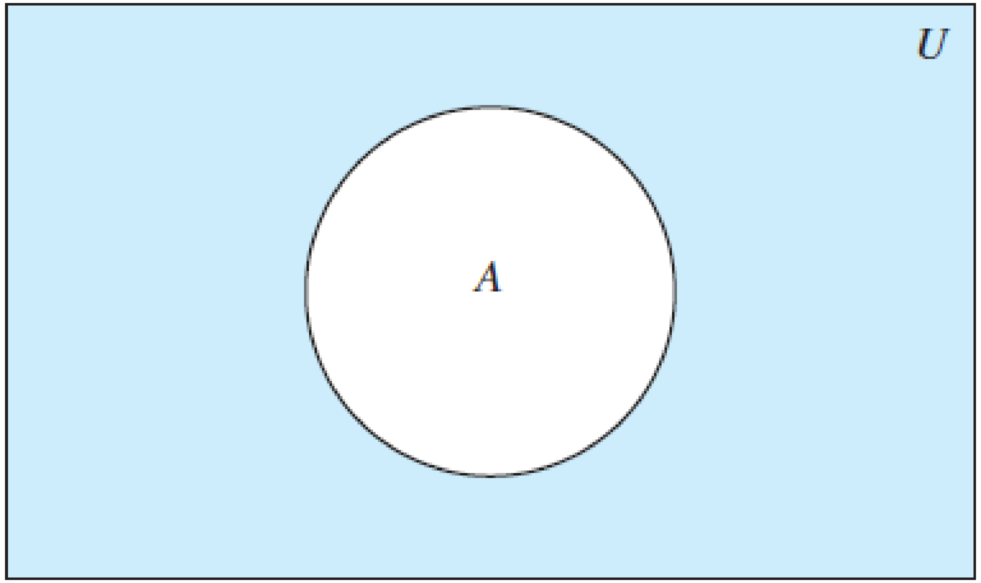
- Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
- $A \cap B = \{x \mid x \in A \land x \in B\}$



$A \cap B$ is shaded.

Complement of a set

- Let U be the universal set. The complement of the set A, denoted by A, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.
- A = {x ∈ U | x ∉ A}



$\overline{A}$ is shaded.

Set identities

Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Computer Representation of Sets

- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance a_1, a_2,…, a_n. Represent a subset A of U with the bit string of length n, where the i-th bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example

- Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?
- The bit string that represents the set of odd integers in U, namely, {1, 3, 5, 7, 9}, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is 10 1010 1010. Similarly, we represent the subset of all even integers in U, namely, {2, 4, 6, 8, 10}, by the string 01 0101 0101.
- The set of all integers in U that do not exceed 5, namely, {1, 2, 3, 4, 5}, is represented by the string 11 1110 0000.

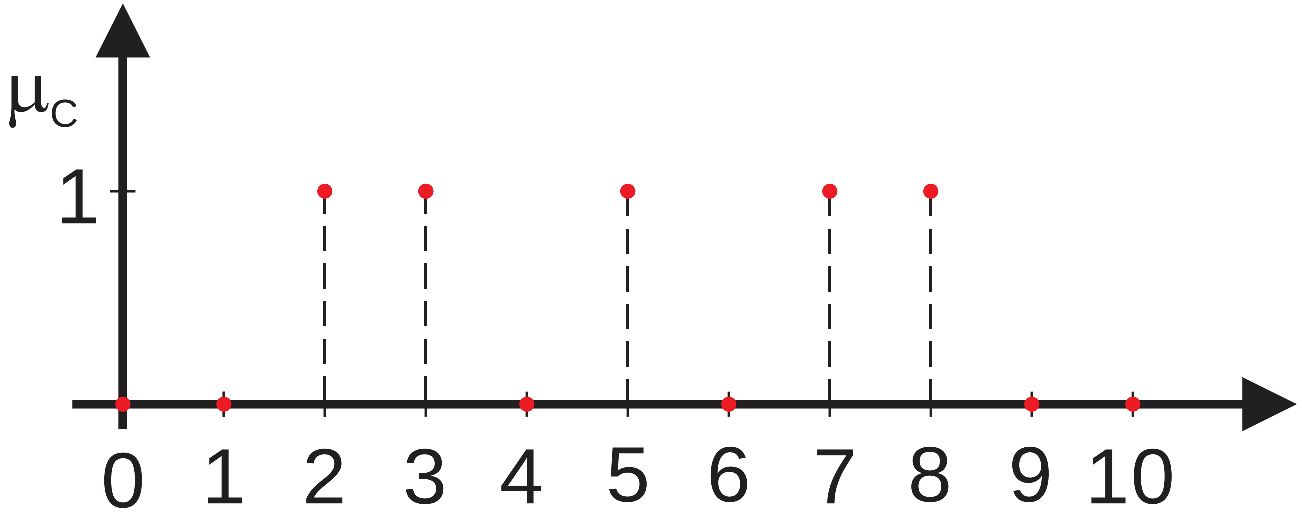
Characteristic function

- $\mu_C : X \rightarrow \{0, 1\}$

- Characteristic function for a certain set A
- $\mu_A(x)=1 \quad \Rightarrow \quad x \in A$
- $\mu_A(x)=0 \quad \Rightarrow \quad x \notin A$

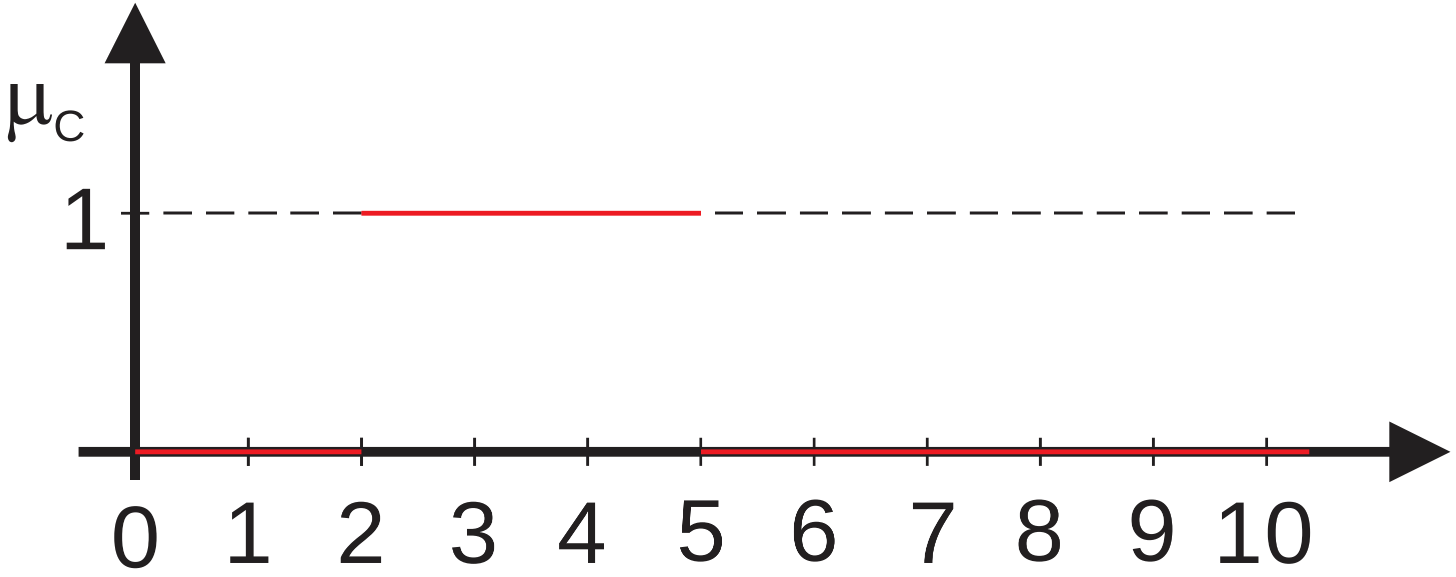
Example - 1

- A={2, 3, 5, 7, 8}



Example - 2

- A={x|2<x<5}



Question

- How would you define intersection, union, etc. using characteristic function? What would be a condition for one set being a subset of another set?