State-space approach for linear control systems - lectures

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Chapter 2. Solution of state equations

Lecture 3: time domain solution of a linear state equation

We start with a comparison to a first order linear ODE to move towards a multi-dimensional case which relies on matrix algebra. Our goal is to build the solution of linear time invariant models expressed in the standard state equation form

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t),$$

$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t) \tag{2.1}$$

First, let's look at a *homogeneous* case which is $\vec{u}(t) = 0$ and $\vec{x}(0) = \vec{x}_0$

$$\dot{\vec{x}} = A\vec{x}. \tag{2.2}$$

A one dimensional case is

$$\dot{x}(t) = ax(t), \tag{2.3}$$

the solution of which is (for $x(0) = x_0$) :

$$x_h(t) = e^{at}x_0 \tag{2.4}$$

The exponent in (2.4) maybe expanded in a power series

$$x_h(t) = \left(1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + ... + \frac{a^kT^k}{k!} + ...\right)x_0 \tag{2.5}$$

This is an infinite power series that converges for all finite time values $t > 0$. It can be shown that in the n-th dimensional case $\dot{\vec{x}}(t) = A\vec{x}(t)$ the homogeneous solution is

$$\vec{x}_h(t) = \left(1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... + \frac{A^kt^k}{k!} + ...\right)\vec{x}_0. \tag{2.6}$$

The similarity of (2.5) and (2.6) leads us to the introduction of the *matrix exponent* for a square $n \times n$ matrix A

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... + \frac{A^kt^k}{k!} + ... \tag{2.7}$$

which itself is a square $n \times n$ matrix. So

$$\vec{x}_h(t) = e^{At}\vec{x}_0, \qquad \dot{\vec{x}} = A\dot{\vec{x}} \tag{2.8}$$

Equation (2.8) is often written in a form

$$\vec{x}_h(t) = \Phi(t)\vec{x}_0 \tag{2.9}$$

Where $\Phi(t) = \exp(At)$ and is called the *state transition matrix* which makes sense since (2.9) is a transition from the initial state $\vec{x}_0$ to the state at the time t which is $\vec{x}(t)$.

Example 2.1 a. Let $\vec{x}_0 = \vec{x}(0) = (2; 3)$ and

$$\begin{cases} \dot{x_1} = -2x_1 + u, \\ \dot{x_2} = x_1 - x_2 \end{cases} \qquad A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

For the state transition matrix we shall write the first free terms

$$\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... + \frac{A^kt^k}{k!} + ... =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} t + \begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix} t^2 + \begin{pmatrix} -8 & 0 \\ 7 & -1 \end{pmatrix} t^3 + ... =$$

$$= \begin{pmatrix} 1 - 2t + \frac{4t^2}{2!} + \frac{-8t^3}{3!} + ... & 0 \\ 1 + t + \frac{-3t^2}{2!} + \frac{7t^3}{3!} + ... & 1 - t + \frac{t^2}{2!} + \frac{-t^3}{3!} + ... \end{pmatrix} \tag{2.10}$$

We encourage the reader to write out e^{-2t}, e^{-t} as in formula (2.5) so we can *recognize* that

$$\Phi(t) = \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t}. \end{pmatrix}$$

As $x_0 = (2; 3)$ and $x_h(t) = \Phi(t)x_0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ 2e^{-t} - 2e^{-2t} + 3e^{-t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix}. \qquad \square$$

The forced state response $u(t) \neq 0$ **:**

The complete response of a first order system $\dot{x}(t) = ax(t) + bu(t)$, can be shown and proven by substitution to be [1]

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{2.11}$$

The factor e^{at} can be excluded from integration since it does not depend on the τ internal variable.

$$x(t) = e^{at}x_0 + e^{at}\int_0^t e^{a\tau}bu(\tau)d\tau \tag{2.12}$$

As for the general and dimensional case the solution is again very similar

$$x(t) = e^{At}\vec{x_0} + \int_0^t e^{A(t-\tau)}B\vec{u}(\tau)d\tau \tag{2.13}$$

or

$$x(t) = e^{At}\vec{x_0} + e^{At}\int_0^t e^{A\tau}B\vec{u}(\tau)d\tau. \tag{2.14}$$

Both (2.13) and (2.14) are equal representations of the complete solution of the state equation $\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$. Note that the matrix inside (2.14) $\exp^{-A\tau} B\vec{u}(t)$ is a multiplication of $(n \times n)(n \times p)(p \times 1)$ matrices and is a $(n \times 1)$ column vector. Like any matrix it undergoes integration *element by element*.

Example 2.1 b. Here $u(t) = 5, t > 0$ using the formula (2.14)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix} + \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t}. \end{pmatrix} \times \int_0^t \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t}. \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} 5d\tau =$$

$$= \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix} + \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t}. \end{pmatrix} \times \begin{pmatrix} \int\limits_0^t 5e^{2\tau} d\tau \\ \int\limits_0^t (5e^{\tau} - 5e^{2\tau}) d\tau \end{pmatrix} =$$

$$= \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix} + \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t}. \end{pmatrix} \times \begin{pmatrix} \frac{5}{2}e^{2t} - \frac{5}{2} \\ -\frac{5}{2}e^{2t} + 5e^{t} - \frac{5}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} - \frac{5}{2}e^{-2t} \\ \frac{5}{2}e^{-2t} - 5e^{-t} + \frac{5}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -e^{-2t} + 5 \\ e^{-2t} + 5 \end{pmatrix}. \quad \square \tag{2.15}$$

The system output response:

The algebraic output $\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$ is a sum of a homogeneous (zero input) plus a forced (zero initial condition) components

$$\vec{y}(t) = \vec{y}_h(t) + \vec{y}_f(t) = Ce^{At}\vec{x}_0 + Ce^{At} \int_0^t e^{A\tau} B\vec{u}(\tau) d\tau + D\vec{u}(t). \tag{2.16}$$

Example 2.1 c. Using the same system let's construct the output, for which we have chosen $y(t) = 2x_1 + x_2$,. Please note, that for formula (2.16) we have already done some calculations in (2.15):

$$e^{At} \int_0^t e^{A\tau} B\vec{u}(\tau) d\tau = \begin{pmatrix} \frac{5}{2} - \frac{5}{2}e^{-2t} \\ \frac{5}{2}e^{-2t} - 5e^{-t} + \frac{5}{2} \end{pmatrix}$$

as well as in example 2.1 a, we have already written

$$e^{At}\vec{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{pmatrix}.$$

Then since $C = (2, 1)$ and $D = 0$ we get

$$y(t) = 2 \cdot 2e^{-2t} + 1(5e^{-t} - 2e^{-2t}) + 2\left(\frac{5}{2} - \frac{5}{2}e^{-2t}\right) + 1\left(\frac{5}{2}e^{-2t} - 5e^{-t} + \frac{5}{2}\right) = \frac{15}{2} - \frac{e^{-2t}}{2}. \quad \square$$

Some properties of the state transition matrix $\Phi(t)$

1) $\Phi(-t) = \Phi^{-1}(t)$ this is how we can calculate $\vec{x}_h$ back in time

$$\vec{x_h}(-t) = \Phi^{-1}(t)\vec{x}(0)$$

2) $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$. If the initial condition is set not at $t = 0$ but rather at $t = t_0$ we get

$$\vec{x_h}(t) = \Phi(t - t_0)\vec{x}(t_0) \tag{2.17}$$

Lecture 4: system eigenvalues and eigenvectors

So far we have calculated the matrix exponent of the state transition matrix using a direct method or formula definition (2.7) as in example 2.1 a. These calculations were quite cumbersome so (2.7) is rarely actually used for applied calculations. There are more than a dozen ways to find e^{At} however for *control* purposes by far the best one is driving the square matrix a to the diagonal form first.

Each square matrix $A_{n\times n}$ has a certain ($\leq n$) number of real or complex numbers $\lambda_i, i \leq n$ which are called eigenvalues. What makes them special is that they come in pairs of $(n \times 1)$ column vectors $\vec{m}_i$ called eigenvectors [2]:

$$A\vec{m}_i = \lambda_i \vec{m}_i = \begin{pmatrix} \lambda_i m_{1i} \\ \lambda_i m_{2i} \\ \vdots \\ \lambda_i m_{ni} \end{pmatrix}. \tag{2.18}$$

So you see that for pairs of (λ_i, m_i) the rather complex matrix multiplication of $A\vec{m}_i$ *comes down to a simple multiplication* of the column m_i by λ_i. Note that if $\vec{m}_i$ is an eigenvector of A then $\alpha\vec{m}_i$ is also an eigenvector so we should pair other $(\lambda_i, \alpha\vec{m}_i)$ where α is a real or complex number but $\alpha \neq 0$.

Example 2.2. Find eigenvalues and eigenvectors of A.

The way eigenvalues can values are found is through a *characteristic equation*

$$\det(\lambda I_{n\times n} - A) = 0 \tag{2.19}$$

$$A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}; \quad \lambda I - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} \lambda + 2 & -1 \\ -2 & \lambda + 3 \end{pmatrix};$$

$$\det(\lambda I - A) = 0, \implies (\lambda + 2)(\lambda + 3) - 2 = 0, \quad \lambda^2 + 5\lambda + 4 = 0 \implies \lambda_1 = -4, \ \lambda_2 = -1.$$

So the eigenvalues are $(-4, -1)$.

The way eigenvectors are found is by substituting λ_i into

$$(\lambda_i I - A)\vec{m}_i = 0. \tag{2.20}$$

For $\lambda_1 = -4$ we get

$$(\lambda_1 I - A)\vec{m}_i = \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}\right] \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = 0,$$

$$\begin{pmatrix} -4 + 2 & -1 \\ -2 & -4 + 3 \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = 0 \implies \begin{cases} -2m_{11} - m_{21} = 0, \\ -2m_{11} - m_{21} = 0 \end{cases}$$

which means that $m_{21} = -2m_{11}$ so all vectors $A\vec{m}_1 = (m_{11}; m_{21})$ with such an equality are going to be an eigenvector for $\lambda_1 = -4$. For example, for $\lambda = -4$ we get

$$\vec{m_1} = \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}, \begin{pmatrix} -10 \\ 20 \end{pmatrix}, \ldots$$

For $\lambda_2 = -1$ we get

$$\lambda_2 I - A = \begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}$$

so (2.19) is

$$\begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = 0,$$

and

$$\begin{cases} m_{11} - m_{21} = 0, \\ -2m_{11} + 2m_{21} = 0 \end{cases}$$

so $m_{21} = m_{11}$ and all such vectors are eigenvectors of A:

$$\vec{m_2} = \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -9 \\ -9 \end{pmatrix}, \begin{pmatrix} 2+3i \\ 2+3i \end{pmatrix}, \ldots \qquad \square$$

We have reviewed this process to present a way to calculate a matrix exponent. It turns out that if we construct a *modal matrix* $M = (m_1 m_2 ... m_n)$ where $m_1, m_2, ..., m_n$ are *arbitrary eigenvectors* corresponding to their eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}; \quad \Phi(t) = e^{At} = M e^{\Lambda t} M^{-1}; \tag{2.21}$$

From (2.20) it is obvious that a homogeneous response of a state equation $\dot{\vec{x}} = A\vec{x}$ is

$$\vec{x}_h(t) = \Phi(t)\vec{x}(0) = M e^{\Lambda t} M^{-1} \vec{x}(0). \tag{2.22}$$

Example 2.3. The continuation of example 1.5 — a rotational electromechanical system: a DC servomotor.

Equations (1.34) and (1.35) completely describe the system state equation. For instance, let's assume $L = 1$ N; $R = 2$ Ohms; $J = 1\ kg\ m^2$; $\mu = 1Nms$ (motor shaft damping); $\alpha = 2N\ m/s$ (torque constant); Then, (1.34) looks like

$$\begin{pmatrix} \theta(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} v(t), \quad y(t) = x_1(t) \tag{2.23}$$

Characteristic equation $\det \lambda I - A) = 0$

$$\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{pmatrix} = \lambda(\lambda(\lambda+3)+2) = \lambda(\lambda^2+3\lambda+2) = 0, \quad \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = -2.$$

Here you can see that one of the eigenvalues is 0, therefore we cannot use formulas (2.18 - 2.20). Luckily there are plenty of other ways to compute a matrix exponent.

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k, \tag{2.24}$$

where scalar analytic functions α satisfy

$$W = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix}, \tag{2.25}$$

where W is a *Vandermonde matrix*. From (2.23) clearly

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = W^{-1} \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{pmatrix} \tag{2.26}$$

For our example

$$W = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{pmatrix}; \qquad W^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1.5 & -2 & 0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix};$$

Using (2.24)

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1.5 & -2 & 0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix} \begin{pmatrix} e^{0t} \\ e^{-t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 - 2e^{-t} + 0.5e^{-2t} \\ 0.5 - e^{-t} + 0.5e^{-2t} \end{pmatrix}$$

Formula (2.22):

$$e^{At} = \Phi(t) = \alpha_0 + \alpha_1 A + \alpha_2 A^2 = 1I + (1.5 - 2e^{-t} + 0.5e^{-2t}) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} +$$

$$+(0.5 - e^{-t} + 0.5e^{-2t}) \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 - 2e^{-t} + 0.5e^{-2t} & 0.5 - e^{-t} + 0.5e^{-2t} \\ 0 & 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 0 & -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}.$$

Let's assume $u(t) = 0$ and $\vec{x}_0 = (1; 1; 1)$. Using this and equation (2.14) we can acquire the zero-input *open loop state response* (Figure 2.1). Figure 2.1 shows how does each of the state variables change over time. Let's look at the way one can do this in Matlab.

```
% Input the matrices.
A_ccf = [ 0 1 0; 0 0 1; 0 -2 -3 ]; B_ccf = [ 0; 0; 2 ];
C_ccf = [1 0 0]; D_ccf = 0;
```

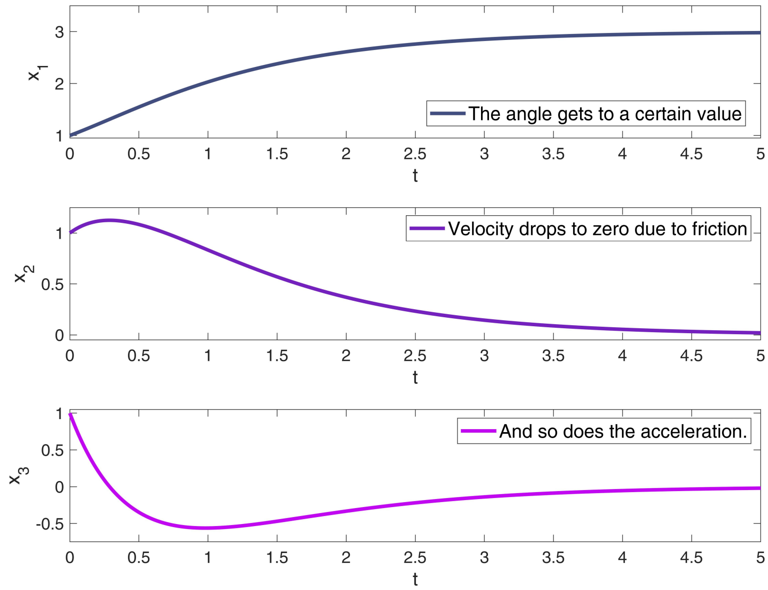

Fig. 2.1: State variables x_1, x_2, x_3 over time.

```
% % Create a state-space representation of the open-loop
% % uncontrolled system.
system = ss(A_ccf,B_ccf,C_ccf,D_ccf);
% % A characteristic polynomial of the open-loop system.
Poly_opl = poly(A_ccf);
% % Eigenvalues - the roots of the characteristic polynomial.
roots(Poly_opl)

% Set up time.
t = 0:0.01:5;
% Set up initial conditions.
X0 = ones(3,1);
% Set up the signal values.
U = zeros(size(t) );

% Get the time and signal data of each of the state variables.
% This function automatically calculates the response using either
% formulas (2.21-2.22) or the Vandermonde matrix approach.
[Y_uncont, t, X_uncont] = lsim( system, U, t, X0 );

% Plot the response.
subplot(311), plot( t,X_uncont(:,1) );
ylim( [0.95 3.5] );
subplot(312), plot( t,X_uncont(:,2) );
ylim( [-0.05 1.25] );
subplot(313), plot( t,X_uncont(:,3) );
ylim( [-0.75 1.05] );
```

References

[1] Williams II, R. L.; Lawrence, D. A. *Linear State-Space Control Systems*, 2nd ed.; John Wiley & Sons: 2007.

[2] Rowell D., Analysis and Design of Feedback Control Systems. Available online: http://web.mit.edu/2.14/www/Handouts/Handouts.html