State-space approach for linear control systems – lectures

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Chapter I. Introduction to the state-space approach

Lecture 1. State-space representation of a control system

An engineering system is often represented as a system of ordinary differential equations (ODEs). The classical approach involves converting this system to a transfer function [1]. Such an operation usually involves a certain integral transform like the Laplace or the Z-transform. This algebraically relates the system's model to the complex integral transformation images of both input and output signals. Replacing a differential equation with an algebraic one simplifies modeling of interconnected systems. However, these methods are mainly applicable to single-input single-output systems, which limits the capabilities of closed-loop feedback control. A variety of engineering processes require a mathematical representation of multiple-input multiple-output systems, which have led to the development of the state-space approach (also known as time-domain approach [2, 3]).

State-space can be used to represent nonlinear systems that have backlash, saturation and dead-zone. Also, it easily handles nonzero initial conditions. This lecture introduces the state-space representation for linear time invariant systems. We show how to derive state equations for physical systems described with a single ODE, a system of them or represented as a transfer function. Let's start with a one-dimensional case of a linear time-invariant system:

$$\begin{cases} \dot{x}(t) = a \cdot x(t) + b \cdot u(t) \\ x(t_0) = x_0 \\ y(t) = c \cdot x(t) + d \cdot u(t). \end{cases} \tag{1.1}$$

The first two expressions of (1.1) is a classical Cauchy problem for an ODE. Let's refer to function $x(t)$ as a **state variable**. The $u(t)$ is the **input** function, while $y(t)$ is the **output**. As we are going to see later, this form of an ODE system is very common with the constant coefficients a, b, c and d describing the relations between the input, output and the so-called state variable.

Example 1.1. The RL network.

Here the one loop of Kirchoff's or Ohm's law (Fig. 1.1) gives

$$L\frac{di}{dt} + Ri = v(t), \#(1.2)$$

where $v(t)$ is the input DC voltage, R is the resistance, L is the inductance and $i = i(t)$ is the current. Let's rewrite the equation as

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}v(t). \#(1.3)$$

Here it is clear that the state variable is $x(t) = i(t)$, while the input is $u(t) = v(t)$.

Now we see that $a = -\frac{R}{L}, b = \frac{1}{L}$ (see Eq. 1.1). For our purposes here the **zero-state condition** $x(t_0) = x_0$ can be arbitrary, for example,

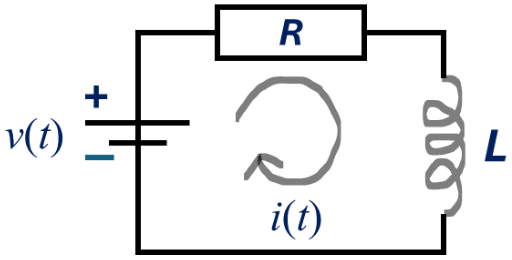


Figure 1.1. RL network

$t_0 = 0$ seconds and $i(0) = 1$ Ampere. As for the output function we have a choice here between $i(t)$, $V_R(t)$ or $V_L(t)$. Later we shall see that choosing the output or even state variables is an important part of the state-space representation. Let's choose $y(t) = i(t)$, then $c = 1$ and $d = 0$ (see Eq. 1.1).

State equations – a general case

A state-space representation of a linear time-invariant system (LTI) has the following general form:

$$\begin{cases} \dot{\vec{x}}(t) = A \cdot \vec{x}(t) + B \cdot \vec{u}(t) \\ \vec{x}(t_0) = \vec{x_0} \\ \vec{y}(t) = C \cdot \vec{x}(t) + D \cdot \vec{u}(t) \end{cases} \#(1.4)$$

This time, however, equations (1.4) are written for vectors and matrices, and $\vec{x}(t)$ is a **state vector** with size $(n \times 1)$:

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix},$$

where $x_1(t), \dots\ x_n(t)$ are functions of time and are the chosen state variables. Same applies to $\dot{\vec{x}}(t)$ and $\vec{x_0}$, both having size $(n \times 1)$:

$$\dot{\vec{x}}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dots \\ \dot{x_n}(t) \end{pmatrix}, \quad \vec{x_0} = \vec{x}(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \dots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ \dots \\ x_{n0} \end{pmatrix},$$

where $\dot{x_1}(t), \dots, \dot{x_n}(t)$ are functions of time and $x_{10}, \dots, x_{n0}$ are constant **initial state values .**

Now let's discuss the matrices A, B, C and D. Since $\dot{\vec{x}}(t)$, $\vec{x}(t)$ and $\vec{x_0}$ are $(n \times 1)$ vectors or columns, A is a **square** $(n \times n)$ matrix:

$$A^{(n \times n)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \#(1.5)$$

where a_{ij} are constant values. Here and later the matrix size will be shown in upper index.

The number of inputs is not necessarily the same as the number of state variables, therefore we assume $\vec{u}(t)$ to be a (r+1) vector, so B is a ($n \times r$) matrix:

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdots \\ u_r(t) \end{pmatrix}, \quad B^{(n\times r)} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & a_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{pmatrix}, \quad \#(1.6)$$

where b_{ij} are constant values

At last let's look at the third expression in (1.4). The number of elements in $\vec{y}(t)$ is chosen by the designer. Considering the dimensions of $\vec{u}(t)$, $\vec{x}(t)$ and due to the rules of matrix multiplication (see below) we get:

$$\vec{y}(t)^{(p\times 1)} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ y_p(t) \end{pmatrix}, \quad C^{(p\times n)} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{pn} \end{pmatrix},$$

$$D^{(p\times r)} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pr} \end{pmatrix}, \#(1.7)$$

where c_{ij} are constant values. Please note that unlike A, the matrix C **may not be** (and usually isn't) **square**, as in general $n \neq p$. Just like we have just shown that C is $(p \times n)$ matrix, we encourage the reader to check and prove that D is a $(p \times r)$ matrix.

<u>Supplementary mathematics – vector and matrix multiplication.</u>

Based on the given number of column elements in $\vec{x}(t)$ and $\vec{u}(t)$ as well as the number of chosen output parameters in $\vec{y}(t)$ in (1.4) we have established the dimensions of the matrices A, B, C and D. However, we feel obliged to provide the reader with additional information on linear algebra, particularly, the matrix multiplication.

A matrix is a two-dimensional array of real and complex numbers:

$\begin{pmatrix} 1 & 3 & -8 \\ 0 & -4 & 2 \end{pmatrix}$ is a (2x3) matrix.

$\begin{pmatrix} 2i & 8 & 9 & -1 \end{pmatrix}$ is a (1x4) matrix. Matrices like this are usually called **row vectors.**

$\begin{pmatrix} -1 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ – (4×1) matrices, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ – (3×1) matrix, they are all **column vectors**.

If we look at the $\vec{x}(t)$ vector in (1.4) we see that it is a (n x 1) matrix (or column vector). Same goes for $\dot{\vec{x}}(t)$ and $\overrightarrow{x_0}$.

You can see that (*n* x *m)* matrix has *n* rows and *m* columns. For example, the *C* matrix in 1.7 is a *p* x *n* matrix. A special case is when the number of rows is equal to the number of columns. In this case the matrix is called **square**. Matrix *A* from (1.5) is a square (*n* x *n*) matrix.

We follow our narrative with a few features and facts.

1. Matrices of the same dimensions can be added and subtracted element by element:

$$\begin{pmatrix}1\\-3\\4\end{pmatrix}+\begin{pmatrix}0\\8\\3\end{pmatrix}=\begin{pmatrix}1\\5\\7\end{pmatrix};\quad \begin{pmatrix}9&5\\-4&0\end{pmatrix}-\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}8&4\\-5&-1\end{pmatrix};$$

$$\begin{pmatrix}7&3\\4&i\\-3i&4\end{pmatrix}+\begin{pmatrix}0&-1\\0&3\\10&8\end{pmatrix}=\begin{pmatrix}7&2\\4&3+i\\10-3i&12\end{pmatrix},\qquad \text{where } i=\sqrt{-1}$$

Matrix addition or subtraction for matrices with different dimensions is undefined:

$$\begin{pmatrix}1&3&10\\0&58&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}="undefined".$$

These "+" and "-" operations follow the same rules as for real numbers. For *A* and *B* both *(m x n)* matrices:

$$\begin{gathered}A\pm B=B\pm A,\\(A\pm B)\pm C=A\pm(B\pm C),\#(1.8)\\A\pm\boldsymbol{O}=A,\end{gathered}$$

where $\boldsymbol{O}$ is a (*m* x *n*) matrix where each element equals to zero.

2. A matrix can be multiplied by a real or complex number and it is defined element-wise:

$$\alpha\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\begin{pmatrix}\alpha\cdot a_{11}&\alpha\cdot a_{12}\\\alpha\cdot a_{21}&\alpha\cdot a_{22}\end{pmatrix},$$

$$\alpha(c_{11}\quad c_{12}\quad c_{13})=(\alpha\cdot c_{11}\quad \alpha\cdot c_{12}\quad \alpha\cdot c_{13}),$$

$$\beta\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}\beta\\0\\2\beta\end{pmatrix}.$$

This operation has the following features:

$$\begin{gathered}(\alpha\beta)A=\alpha(\beta A),\\\alpha(A+B)=\alpha A+\alpha B,\#(1.9)\\(\alpha+\beta)A=\alpha A+\beta A\\\boldsymbol{O}\cdot A=\boldsymbol{O},\end{gathered}$$

where α and β – real or complex numbers.

3. The matrix product $C = AB$ of a $(n \text{ x } m)$ matrix A and $(p \text{ x } q)$ matrix B can be defined only when $m = p$. This means that the number of **columns** of A must be equal to the number of **rows** of B. Then $C = AB$:

$$C = [c_{ij}], \qquad c_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}, \#(1.10)$$

where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Expression (1.10) is commonly described by the "row by column" rule. Let's say we multiply two matrices:

$$C = A^{(3\times2)} \times B^{(2\times4)} = \begin{pmatrix} 1 & 4 \\ 5 & -2 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 8 & 1 \end{pmatrix}$$

The result matrix C is going to be of the size $(3 \times 2) \cdot (2 \times 4) \Rightarrow (3 \times 4)$. The first element $C_{m} = 1 \cdot 1 + 4 \cdot (-1) = -3$ is similar to a scalar product of the first row of A with the first column of B. Just like that C_{34} is a product of row 3 of A and column 4 of B:

$$C_{34} = 3 \cdot 4 - 1 \cdot 1 = 11$$

and

$$C = \begin{pmatrix} 1 \cdot 1 - 4 & 1 \cdot 2 + 4 \cdot 3 & 1 \cdot 3 + 4 \cdot 8 & 1 \cdot 4 + 4 \cdot 1 \\ 5 \cdot 1 + 2 & 5 \cdot 2 - 6 & 5 \cdot 3 - 16 & 5 \cdot 4 - 2 \\ 3 \cdot 1 + 1 & 3 \cdot 2 - 3 & 3 \cdot 3 - 8 & 3 \cdot 4 - 1 \end{pmatrix} = \begin{pmatrix} -3 & 14 & 35 & 8 \\ 7 & 4 & -1 & 18 \\ 4 & 3 & 1 & 11 \end{pmatrix}.$$

The main features of matrix multiplication are:

$$(AB)C = A(BC),$$
$$\text{A}A(B + C) = AB + AC,$$
$$(B + C)A = BA + CA,$$
Generally speaking, $AB \neq BA$.

We can understand the reason why is matrix multiplication defined the way it is as in (1.10) by a closer look at equations (1.4):

$$\begin{cases} \dot{x_1} = a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n + b_{11} \cdot u_1 + \cdots + b_{1r} \cdot u_r \\ \dot{x_2} = a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n + b_{21} \cdot u_1 + \cdots + b_{2r} \cdot u_r \\ \vdots \\ \dot{x_n} = a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \cdots + a_{nn} \cdot x_n + b_{n1} \cdot u_1 + \cdots + b_{nr} \cdot u_r \end{cases} . \#(1.11)$$

The reason why systems like (1.11) are called linear is because each of the $\dot{x_k}$, $k = 1, 2, \dots, n$ depends on a **linear combination** of the state variables $x_1, x_2, \dots, x_n$, since all of the a_{ij} are constant values (compare with a linear function $z(t) = \alpha \cdot t + \beta$, where α and β are real numbers). System (1.11) lets us understand the definition (1.10) since

$$\dot{\vec{x}}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \cdots \\ \dot{x_n}(t) \end{pmatrix} = A \cdot \vec{x}(t) + B \cdot \vec{u}(t).$$

Same goes for the output $\vec{y}(t)$:

$$\vec{y}(t) = C \cdot \vec{x}(t) + D \cdot \vec{u}(t),$$

$$\begin{cases} y_1 = c_{11} \cdot x_1 + c \cdot x_2 + \dots + c_{1n} \cdot x_n + d_{11} \cdot u_1 + \dots + d_{1r} \cdot u_r \\ y_2 = c_{21} \cdot x_1 + c_{22} \cdot x_2 + \dots + c_{2n} \cdot x_n + d_{21} \cdot u_1 + \dots + d_{2r} \cdot u_r \\ \vdots \\ y_p = c_{n1} \cdot x_1 + c_{n2} \cdot x_2 + \dots + c_{nn} \cdot x_n + d_{n1} \cdot u_1 + \dots + d_{nr} \cdot u_r \end{cases} . \#(1.12)$$

Example 1.2. A simple translational mechanical system

Figure 1.2 depicts the mass *m* being under the influence of three forces: the external input force **F**(t) and two resistance forces: spring load $\mathbf{F}_S$ and the viscous damping force $\mathbf{F}_V$. We note *k* as the

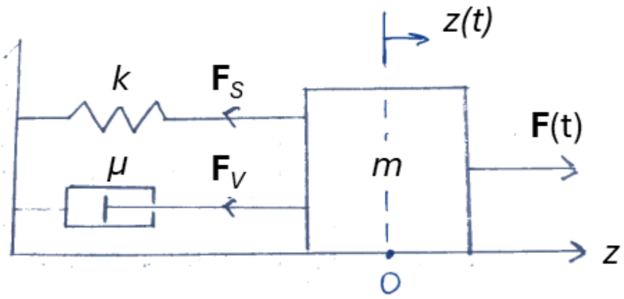


Figure 1.2. Simple translational mechanical system.

spring elasticity coefficient and μ as the viscous friction coefficient of the damper.

Using Newton's second law let's write the dynamic force balance:

$$m \cdot \ddot{z}(t) = f(t) - \mu\dot{z}(t) - kz(t). \#(1.14)$$

Since the highest derivative is by order of 2, we are going to select two state variables.

NB! At large it is a good idea to choose variables connected to the system's energy. The potential energy is $U = \frac{1}{2}k \cdot z(t)^2$ and kinetic energy is $T = \frac{1}{2}m \cdot \dot{z}(t)^2$, so we choose:

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \begin{matrix} x_1(t) = z(t) \\ x_2(t) = \dot{z}(t) = \dot{x_1}(t) \end{matrix},$$

then $\ddot{y}(t) = \dot{x_2}(t)$ and (1.14) is

$$m\dot{x_2}(t) = f(t) - \mu x_2(t) - kx_1(t) \#(1.15)$$

Now let's combine all we have keeping in mind that we are constructing the form $\dot{\vec{x}}(t) = A \cdot \vec{x}(t) + B \cdot \vec{u}(t)$. Here $\vec{u}(t) = f(t)$ is a one-element matrix(vector):

$$\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -\frac{k}{m}x_1(t) - \frac{\mu}{m}x_2(t) + \frac{f(t)}{m} \end{cases}, \#(1.16)$$

so $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\mu}{m} \end{pmatrix}, B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$.

As for the output in this system there is a choice, however the most obvious one is the mass displacement $y(t) = z(t) = x_1(t)$. Since $\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$ we see that $C = (1 \quad 0)$ and $D = 0$:

$$\begin{cases} \vec{y}(t) = x_1(t) = (1 \quad 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \\ \dot{\vec{x}}(t) = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{\mu}{m} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \end{cases},$$

which is the (1.1) classic **state space** form. Also, $B = 0$.

Lecture 2. Examples of the state-space representation

Example 1.3. An electrical network with two inputs

The two inputs are the independent voltage and current sources $V_{in}(t)$ and $i_{in}(t)$ (Figure 2.1):

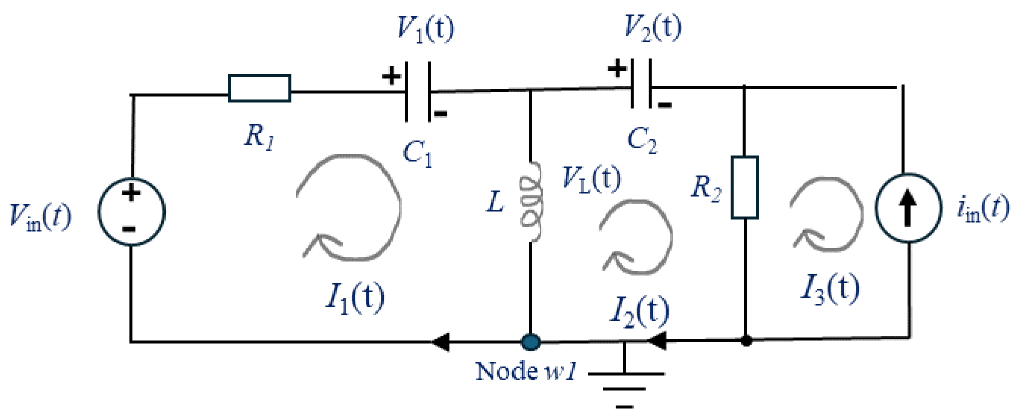


Figure 2.1. Electrical network with two inputs

We assume clockwise circulating currents in each of the three meshes. The output is chosen to be the inductor voltage: $V_L(t)$. First we apply Kirchoff's voltage and current laws to the two leftmost meshes and the node $w1$:

$$V_{in}(t) = I_1(t) \cdot R_1 + V_1(t) + L\frac{d}{dt}\big(I_1(t) - I_2(t)\big);$$

$$\text{Node } w1\text{: } I_L(t) + I_2(t) - I_1(t) = 0.$$

Then

$$V_2(t) + \big(I_2(t) - I_3(t)\big) \cdot R_2 + L\frac{d}{dt}\big(I_2(t) - I_1(t)\big) = 0.$$

Since $i_{in}(t)$ is a current source, $I_3(t) = i_{in}(t)$.

NB! The energy in this circuit is stored in the capacitors and the inductor, so it is convenient to choose the following state variables:

$$\begin{cases} x_1(t) = V_1(t) \\ x_2(t) = V_2(t) \\ x_3(t) = I_L(t). \end{cases} \quad (1.17)$$

The inductor current $I_L(t)$ can be expressed as $I_L(t) = I_2(t) - I_1(t)$. The inputs are the sources:

$$\begin{cases} U_1(t) = V_{in}(t), \\ U_2(t) = i_{in}(t) \end{cases} \#(1.18)$$

The output:

$$y(t) = V_L(t) = L\frac{dI_L}{dt} = L\dot{x}_3 \#(1.19)$$

For a capacitor $I = C\frac{dV}{dt}$, so

$$\begin{cases} I_1(t) = C_1 \cdot V_1(t) = C_1\dot{x}_1 \\ I_2(t) = C_2\dot{x}_2 \\ I_L(t) = I_2(t) - I_1(t) = C_1\dot{x}_1 - C_2\dot{x}_2 \end{cases} \#(1.20)$$

NB! We write it this way so that all of the variables in the Kirchoff's equations are expressed through state variables x_1, x_2, x_3. The energy of the charge stored in a capacitor is $W_c = \frac{1}{2}C{V_c}^2$, the energy of the inductor current flow $W_L = \frac{1}{2}L{I_L}^2$. Now the Kirchoff equations can be recast as[1]:

$$\begin{cases} C_1\dot{x}_1 - C_2\dot{x}_2 = x_3(t), \\ V_1(t) = C_1\dot{x}_1R_1 + x_1(t) - L\dot{x}_3, \\ x_2(t) + \left(C_2\dot{x}_2 - V_2(t)\right)R_2 + L(C_1\dot{x}_1 - C_2\dot{x}_2) = 0. \end{cases}$$

Now rearrange this so that the terms with $\dot{x}_1,\ \dot{x}_2,\ \dot{x}_3$ are on the left side:

$$\begin{cases} C_1\dot{x}_1 - C_2\dot{x}_2 = x_3(t), \\ C_1R_1\dot{x}_1 - L\dot{x}_3 = V_1(t) - x_1(t), \\ LC_1\dot{x}_1 + (R_2C_2 - LC_2)\dot{x}_2 = R_2V_2(t) - x_2(t). \end{cases} \#(1.21)$$

You can see that this is not yet the form we are looking for which is $\dot{\vec{x}}(t) = A \cdot \vec{x}(t) + B \cdot \vec{u}(t)$ but rather

$$\begin{pmatrix} R_1C_1 & 0 & -L \\ LC_1 & R_2C_2 - LC_2 & 0 \\ C_1 & -C_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix}. \#(1.22)$$

NB! Equation (1.22) can be considered as

$$E_{(3x3)}\dot{\vec{x}}(t) = F_{(3x3)} \cdot \vec{x}(t) + G_{(3x2)} \cdot \vec{u}(t) \Big| \times E^{-1}\ on\ the\ left$$

Here we multiply the equation by E^{-1} this is called an **inverse** matrix. Inverse matrices are defined for square matrices. If T is a $(n\times n)$ matrix the inverse matrix T^1 is such that $T \cdot T^1 = I_{n\times n}$, or a $T^1 \cdot T = I_{n\times n}$, where $I_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is a diagonal unit square $(n\times n)$ matrix.

Definition. Each square matrix T has a T^1 as long as its determinant det $T \neq 0$. Then the matrix is called **non-singular**. We will discuss determinants more in depth later. For now we just show two examples:

[1] Note that for convenience we shall sometimes write $\dot{x}_1$ instead $\dot{x}_1(t)$ as well as occasionally x_1 instead $x_1(t)$.

$$\det\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = 1\cdot 4 - 2\cdot 3 = -2 \neq 0 \text{ and is non-singular;}$$

$$\det\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} = 1\cdot(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7) = 0 \text{ and is singular.}$$

Also notice that since in general $AB \neq BA$ (see Supplementary mathematics 1), it is important to specify which side do we perform a matrix multiplication. Here we get

$$E^{-1}\cdot E\,\dot{\vec{x}}(t) = E^{-1}\cdot F\cdot\,\vec{x}(t) + E^{-1}\cdot G\cdot\vec{u}(t).$$

Since $E^{-1}E = I_{n\times n}$ and $I_{n\times n}\,\dot{\vec{x}}(t) = \dot{\vec{x}}(t)$

$$\dot{\vec{x}}(t) = E^{-1}\cdot F\cdot\vec{x}(t) + E^{-1}\cdot G\cdot\vec{u}(t), \qquad \#(1.23)$$

$$E^{-1}\cdot F = A^*, \quad E^{-1}\cdot G = B^*,$$

which is the form of (1.1). Let's put this into perspective and give R_1, R_2, C_1, C_2 and L actual values, so we may calculate these matrices. Let R_1 = 250 Ohms, R_2 = 300 Ohms, C_1=20 nF, C_2=40 nF and L = 0,5 mH. Then

$$E = \begin{pmatrix}5\cdot 10^{-6} & 0 & -5\cdot 10^{-4}\\ 10^{-11} & 11{,}99\cdot 10^{-6} & 0\\ 2\cdot 10^{-8} & 4\cdot 10^{-8} & 0\end{pmatrix} = 10^{-6}\begin{pmatrix}5 & 0 & -500\\ 10^{-5} & 11{,}99 & 0\\ 0{,}03 & 0{,}04 & 0\end{pmatrix}$$

$$E^{-1} = 10^{-6}\begin{pmatrix}5 & 0 & -500\\ 10^{-5} & 11{,}99 & 0\\ 0{,}03 & 0{,}04 & 0\end{pmatrix}^{-1} = 10^{6}\cdot(E^*)^{-1}\,\mathit{for\ convenience}.$$

Let's find E^{-1} in **Matlab**. Please, see how to write matrices in Matlab:

```
>>: E_ast = [5  0  -500; 10^-5 11,99  0;  0,02  0,04  0;]
>>: E_ast_minus_one = inv(E_ast); %calculates the inverse matrix
F = [-1 0 0; 0 -1 0; 0 0 1];
G = [1 0; 0 1; 0 0];
A_ast = E_ast_minus_one * F;
B_ast = E_ast_minus_one * G;
```

With these calculations, we end up with

$$E^{-1} \approx 10^{-6}\cdot\begin{pmatrix}0 & -0{,}17 & 50\\ 0 & 0{,}083 & -4{,}17\cdot 10^{-5}\\ -0{,}002 & -0{,}0017 & 0{,}5\end{pmatrix};$$

$$A^* \approx 10^{-6}\cdot\begin{pmatrix}0 & 0{,}17 & 50\\ 0 & -0{,}083 & -4{,}17\cdot 10^{-5}\\ 0{,}002 & 0{,}0017 & 0{,}5\end{pmatrix},\ B^* = \begin{pmatrix}0 & -0{,}17\\ 0 & 0{,}083\\ -0{,}002 & -0{,}0017\end{pmatrix};$$

So, $\dot{\vec{x}} = 10^{-6}\cdot A^*\,\vec{x}(t) + 10^{-6}\cdot B^*\,\vec{u}(t)$

$$\dot{\vec{x}}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = 10^{-6} \cdot \begin{pmatrix} 0 & 0{,}17 & 50 \\ 0 & -0{,}083 & -4{,}17 \cdot 10^{-5} \\ 0{,}002 & 0{,}0017 & 0{,}5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \\ +10^{-6} \cdot \begin{pmatrix} 0 & -0{,}17 \\ 0 & 0{,}083 \\ -0{,}002 & -0{,}0017 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Ax + Bu; \#(1.24)$$

The output that we want to see (the inductor voltage $V_L = L\dot{x}_3$) is

$$\vec{y}(t) = 10^{-6}(0{,}002 \quad 0{,}0027 \quad 0{,}5) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \\ 10^{-6}(-0{,}002 - 0{,}0017) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Cx + Du; \#(1.25)$$

which should be obvious since we take only the last rows in (1.24) to get $\dot{x}_3(t)$**.** Expressions (1.24) and (1.25) complete the state space representation. This example shows that sometimes simple matrix arithmetic is required in order to make a state-space representation unlike example 1.2.

NB! Please note that the number of state variables is tied to the number and order of the given equations. Example 1.2 has a simple second-order differential equation, therefore has two first order differential equations (voltage law) and algebraic one (the node currents), therefore it has three state variables x_1, x_2 and x_3.

Example 1.4. A single-input, single-output rotational mechanical system.

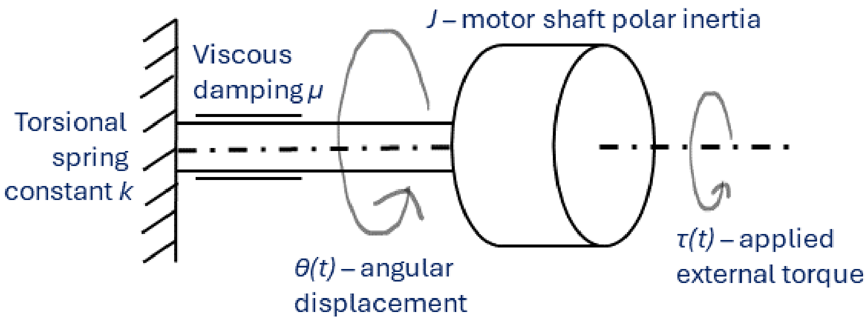


Figure 1.4 A single-input, single-output rotational mechanical system

If the shaft is flexible $k \neq 0$. Both the viscous damping and the spring reaction by twisting the shaft at an angle $\theta(t)$ oppose the external torque $\tau(t)$. For rotational mechanical systems there is the equivalent of the Newton's second law called **Euler rotational law**:

$$J\ddot{\theta}(t) = \tau(t) - \mu \cdot \dot{\theta}(t) - k \cdot \theta(t) \#(1.26)$$

This is a single second order ODE so we have two state variables to choose. Once again, the energy storage of $W = \frac{1}{2}k \cdot \theta(t)^2$and $T = \frac{1}{2}J \cdot \dot{\theta}(t)^2$ hint us for

$$x_1(t) = \theta(t),$$
$$x_2(t) = \dot{\theta}(t) = \dot{x}_1$$

Then (1.26) is written as $J\dot{x}_2 = -kx_1 - \mu x_2 + \tau(t)$, so

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\frac{k}{J}x_1 - \frac{\mu}{J}x_2 + \frac{\tau(t)}{J} \end{cases}. \#(1.27)$$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{J} & -\frac{\mu}{J} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{J} \end{pmatrix} \tau(t). \#(1.28)$$

For the output let's choose the angular displacement $y(t) = \theta(t) = x_1$

$$y(t) = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [0] \cdot \tau(t). \#(1.29)$$

So, we have *A*, *B*, *C* and *D*.

Example 1.5. Rotational Electromechanical System: a DC servomotor

The input is armature voltage *v*(*t*) and the output is the motor shaft angular displacement. *L*, *R*, μ and *J* are constants with *J* being the motor shaft polar inertia. Also here we ignore back emf voltage. Angular velocity is $\omega(t) = \dot{\theta}(t)$. The rest goes like this:

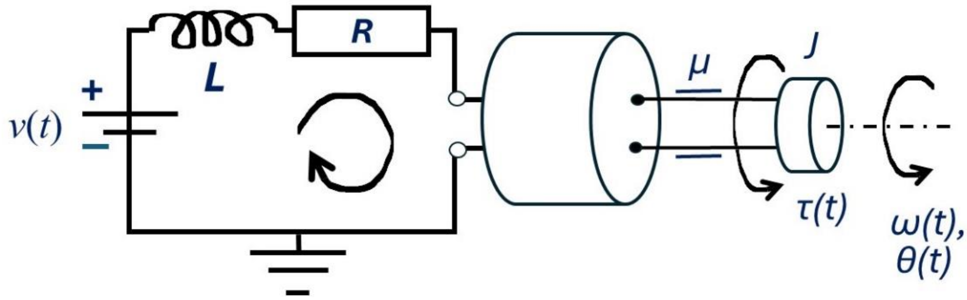


Figure 1.5 Rotational electromechanical system: a DC servomotor

1) Electrical circuit model: the Kirchoff's voltage law:

$$L\frac{di}{dt} + R \cdot i(t) = v(t) \#(1.30)$$

2) Electromechanical coupling:

$$\tau(t) = \alpha \cdot i(t) \#(1.31)$$

So, the torque is linearly dependent on current , where α is the motor torque constant.

3) Rotational mechanical model:

$$J\ddot{\theta}(t) + \mu \cdot \dot{\theta}(t) = \tau(t). \#(1.32)$$

NB! Here it is convenient to use a block diagram and the Laplace transform.
If we assume zero initial current

$$\frac{di}{dt} \leftrightarrow s \cdot I(s) \; and \; L \cdot s \cdot I(s) + RI(s) = V(s)$$

$\tau(s), \theta(s), V(s)$and $I(s)$ are the Laplace images of corresponding physical quantities.

Let us set T_1 as electrical part of the circuit transfer function, T_2 as electromechanical coupling transfer function and T_3 as rotational mechanical part transfer function
Therefore

$$T_1(s) = \frac{output}{input} = \frac{I(s)}{V(s)} = \frac{1}{Ls + R};$$

From $\tau(t) = \alpha \cdot i(t) \leftrightarrow \tau(s) = \alpha \cdot I(s)$ and

$$T_2(s) = \frac{\tau(s)}{I(s)} = \alpha;$$

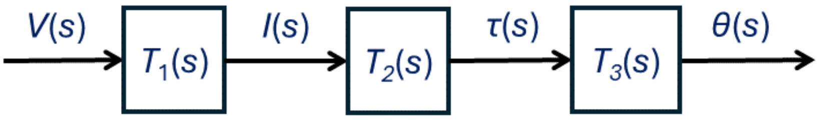
From (1.32) we get

$$Js^2\theta(s) + \mu s\theta(s) = \tau(s)$$

and

$$T_3(s) = \frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2 + \mu s}.$$

A block diagram for the set of employed Laplace transforms:



It shows that in conclusion $T(s) = T_1(s) \cdot T_2(s) \cdot T_3(s)$.

$$T(s) = \frac{\theta(s)}{V(s)} = \frac{\alpha}{(Ls + R)(Js^2 + \mu s)} = \frac{\alpha}{LJs^3 + (L\mu + RJ)s^2 + R\mu s}.$$

Here we use the notation that

$$LJs^3\theta(s) + (L\mu + RJ)s^2\theta(s) + R\mu s\theta(s) = \alpha V(s)$$

is a Laplace image of the following ODE:

$$LJ\dddot{\theta}(t) + (L\mu + RJ)\ddot{\theta}(t) + R\mu\dot{\theta}(t) = \alpha V(t). \#(1.33)$$

Since it is an order of 3 ODE we need 3 state variables.
Now we can choose

$$x_1(t) = \theta(t),$$
$$x_2(t) = \dot{\theta}(t) = \dot{x}_1$$
$$x_3(t) = \ddot{\theta}(t) = \dot{x}_2$$

and

$$\dot{x}_3 = \dddot{\theta}(t) = -\frac{R\mu}{L \cdot J}x_2 - \frac{L\mu + RJ}{L \cdot J}x_3 + \frac{\alpha}{L \cdot J}V(t).$$

Thus

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{R\mu}{L \cdot J} & -\frac{L\mu + RJ}{L \cdot J} & -\frac{L\mu + RJ}{L \cdot J} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha}{L \cdot J} \end{pmatrix} \cdot V(t), \#(1.34)$$

We care only about $\theta(t)$, so

$$y(t) = \theta(t) = x_1(t),$$
$$y(t) = (1 \quad 0 \quad 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 0 \cdot V(t), \#(1.35)$$

which completes the state space representation of the DC motor.

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3. Rowell D., Analysis and Design of Feedback Control Systems. Available online: http://web.mit.edu/2.14/www/Handouts/Handouts.html